

# Cargèse Lectures on String Theory with Eight Supercharges

Monica Guica and Andrew Strominger  
 Jefferson Physical Laboratory  
 Harvard University  
 Cambridge, MA 02138

These lectures give an introduction to the interrelated topics of Calabi-Yau compactification of the type II string, black hole attractors, the all-orders entropy formula, the dual  $(0, 4)$  CFT, topological strings and the OSV conjecture. Based on notes by MG of lectures by AS at the 2006 Cargèse summer school.

## Contents

<b>1 Introduction</b>	<b>1</b>	<b>5 <math>Z_{\text{BH}}</math>, <math>Z_{\text{CFT}}</math>, <math>Z_{\text{IIA}}</math>, <math>Z_{\text{top}}</math> and all that</b>	<b>26</b>
		5.1 Definitions . . . . .	26
		5.2 Relations . . . . .	27
<b>2 BPS Calabi-Yau black holes</b>	<b>3</b>	<b>1. Introduction</b>	
2.1 IIB Calabi-Yau compactification . . . . .	3	One of the best understood quantum systems	
2.2 IIA Calabi-Yau compactification . . . . .	5	is IIB string theory on $AdS_5 \times S^5$ with its dual	
2.3 $\mathcal{N} = 2$ 4d supergravity and special		presentation as $\mathcal{N} = 4$ Yang Mills gauge theory.	
geometry . . . . .	5	The tractability of this system is in large part due	
2.4 Black hole solutions . . . . .	7	to the large supersymmetry group which has the	
2.5 The attractor equations . . . . .	8	maximal 32 supercharges. But a large symmetry	
2.5.1 A simple example . . . . .	10	group is both a blessing and a curse. It is a bless-	
2.6 The general asymptotically flat so-		ing because many features of the theory can be	
lution . . . . .	10	deduced using symmetry arguments alone. It is	
2.6.1 Simple example redux . . . . .	11	a curse because these same symmetries limit the	
2.6.2 Generic entropy for cubic		possible dynamics and questions we can ask. For	
prepotential . . . . .	11	example, true tests of the various dualities which	
2.7 Higher orders . . . . .	12	are not already implied by the symmetries of the	
2.7.1 Corrections to the $\mathcal{N} = 2$		32-supercharge theory are relatively hard to find.	
action . . . . .	12	As the number of supersymmetries is de-	
2.7.2 Wald's formula . . . . .	13	creased, more dynamical possibilities are un-	
2.8 The MSW CFT . . . . .	13	leashed, often with no counterparts in their more	
2.9 The (modified) elliptic genus . . . . .	15	symmetric cousins. Ultimately the most interest-	
<b>3 The topological string</b>	<b>17</b>	ing and most physically relevant case is no su-	
3.1 Twisting the string . . . . .	17	persymmetry at all. But at the moment analytic	
3.2 The A-model topological string . . . . .	18	control in this case is quite limited. At the current	
<b>4 The OSV conjecture</b>	<b>19</b>	juncture, the intermediate case of eight super-	
4.1 Mixed partition functions . . . . .	20	charges has proven to be especially fertile ground	
4.2 Brute force derivation of OSV . . . . .	21	for progress. Eight supercharges is where black	
4.3 Why $Z_{\text{BH}} =  Z_{\text{top}} ^2$ . . . . .	22	holes are first encountered <sup>1</sup> , as well as Calabi-	
4.3.1 $M$ -theory lift . . . . .	22		
4.3.2 Computing $Z_{\text{IIA}}$ . . . . .	24		

<sup>1</sup>Of course, black holes exist in compactifications with more supersymmetry, but the near-horizon geometry has at most eight supersymmetries in all known examples.

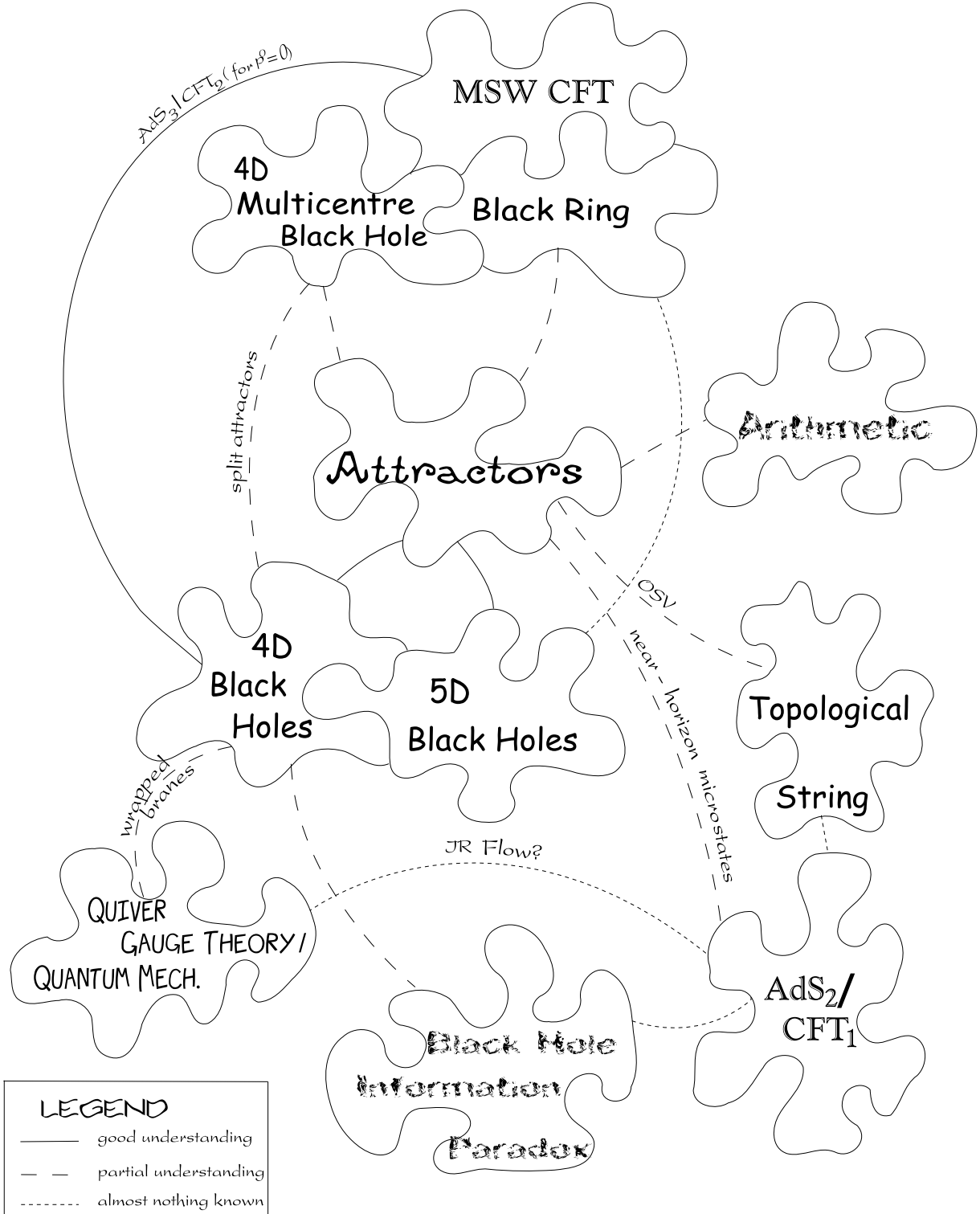


Figure 1. The eight-supercharge jigsaw puzzle. It probably cannot be assembled on the plane.

Yau spaces and topological strings. Despite several decades of work in this area, qualitative new surprises are still appearing (e.g. [1–4]). It is likely that a few more will still appear before the eight-supercharge jigsaw puzzle, illustrated in figure 1, is pieced together. For example it has been suggested [5,6] that number theory, which has yet to be fully wed to string theory, will play a prominent role. Another likely suspect, as yet to fully appear in the story, is the enigmatic  $AdS_2/CFT_1$  duality.

Among the fascinating connections illustrated in figure 1, these lectures will focus on those directly relevant to understanding the so-called OSV conjecture [3]. The agreement between the Bekenstein-Hawking area law and the microscopic state counting for certain stringy black holes with eight near-horizon supercharges has been the source of many deep insights into string theory and quantum gravity. But this area law is just the first term in an infinite series approximating the exact expression for black hole entropy. OSV conjecturally relates every term in this series to expansion coefficients of the topological string, thereby potentially allowing for “precision tests” of holographic duality. We are likely to have much to learn by precisely formulating, proving or disproving OSV.

In these lectures we will present some introductory material on string theory with eight supercharges, aimed at the student preparing for research in this area, and building up to a discussion of the OSV conjecture in the last section. For excellent and recent reviews at a more advanced level see [7,8]. We begin in lecture 2 with the basics of Type II Calabi-Yau compactification, BPS black holes, attractors, and their CFT duals. In lecture 3 we give a brief introduction to the topological string. In lecture 4 we describe the OSV conjecture which relates lectures 2 and 3. In lecture 5 we conclude by summarizing the various partition functions we have encountered along the way, the extent to which they are well defined, and the relations between them.

## 2. BPS Calabi-Yau black holes

In this lecture we will briefly review the field content of type IIA/IIB string theory compactified on a Calabi-Yau manifold, and the resulting  $\mathcal{N} = 2$  4d supergravity action. We will show that the 4d theory admits supersymmetric black hole solutions of the extremal Reissner-Nordström type, which exhibit an interesting attractor mechanism for the vector moduli as one approaches the horizon of the black hole. The macroscopic entropy of these black holes is computed and it is shown how to include higher order corrections. We briefly discuss the proposed microscopic dual, known as the MSW CFT, for the special case of no  $D6$  charge. (Finding the dual for the general case is an important unsolved problem). We also describe the index - the modified elliptic genus - which counts the weighted number of BPS states in this CFT.

Throughout these lectures, we will use whichever of the type IIA/IIB languages makes the exposition simplest. Thus the 4d  $\mathcal{N} = 2$  action will be discussed mostly in the context of IIB compactifications, while in order to talk about black holes we will switch to the IIA description, which is also the one appropriate for understanding the OSV conjecture.

### 2.1. IIB Calabi-Yau compactification

The bosonic 10d field content of type IIB string theory is as follows: from the NS-NS sector we get the graviton  $G_{MN}$ , the antisymmetric tensor field  $B_{MN}$  with field strength  $H^{(3)} = dB^{(2)}$ , and the dilaton  $\Phi$ . From the R-R sector we have a scalar  $a$  (the axion), a two-form with  $F^{(3)} = dC^{(2)}$  and a self-dual four-form obeying (at the linearized level)  $F^{(5)} = dC^{(4)} = \star F^{(5)}$ .

Compactifying six of the ten dimensions on a Calabi-Yau manifold  $M$  breaks three quarters of the original 32 component supersymmetries of type IIB supergravity. Since in four dimensions the minimal spinors have four components, the eight supersymmetries we are left with give  $\mathcal{N} = 2$  supergravity in four dimensions. This theory - described in great detail in the fundamental paper [9] - has an  $SU(2)_R$   $R$ -symmetry under which the two supercharges transform as

doublets. The massless fields fall into  $\mathcal{N} = 2$  representations labeled by their highest spins. Three kinds of multiplets appear:

- **gravity multiplet:** contains the graviton (2), two gravitini ( $\frac{3}{2}$ ) in an  $SU(2)_R$  doublet and the graviphoton (1),
- **vector multiplets:** contain one photon (1), two fermions ( $\frac{1}{2}$ ) in an  $SU(2)_R$  doublet and two real scalars (0),
- **hypermultiplets:** contain two hyperfermions ( $\frac{1}{2}$ ) and four hyperscalars (0) in two  $SU(2)_R$  doublets,

where in parentheses we have written the spins of the corresponding particles.

Now let us figure out how many multiplets of each kind we get from the compactification<sup>2</sup>. Upon dimensional reduction from  $10d$  to  $4d$ , the massless wave equation splits into two pieces, schematically  $\square_{10} = \square_4 + \square_{CY}$ . Harmonic forms  $\omega$  satisfy  $\square_{CY}\omega = 0$  and they are in one-to-one correspondence with cohomology classes on  $M$ , whose numbers are counted by the Hodge numbers of  $M$ ,  $h_{p,q}$ <sup>3</sup>. Thus harmonic forms on  $M$  lead to massless fields in  $4d$ .

There are two multiplets that we obtain in *any* compactification of type IIB on a Calabi-Yau: the gravity multiplet and the so-called *universal hypermultiplet*. The universal hypermultiplet consists of the  $10d$  dilaton  $\Phi$ , the  $10d$  axion  $a$ , plus two more massless scalars  $\chi$  and  $\psi$  defined by writing  $H^{(3)} = \star_4 d\psi^4$  and  $F^{(3)} = \star_4 d\chi$ . The bosonic content of the universal hypermultiplet is then  $(\Phi, a, \psi, \chi)$  and it will obviously be the same no matter which Calabi-Yau we choose to compactify on. Importantly, the string coupling

<sup>2</sup>Here we will only follow what happens to the massless bosons that arise from the compactification; the fermions complete the supermultiplets.

<sup>3</sup> $h_{p,q}$  is the number of harmonic forms of antiholomorphic rank  $p$  and holomorphic rank  $q$ . A Calabi-Yau manifold has  $h_{0,0} = 1$ ,  $h_{0,i} = 0$  for  $i \neq 3$ , one harmonic (0,3) form  $\Omega$  and its (3,0) conjugate  $\bar{\Omega}$ , and  $h_{1,1} \geq 1$ ,  $h_{1,2} \geq 0$ . All other Hodge numbers are given by Poincaré duality  $h_{p,q} = h_{3-p,3-q}$ . There is a total of  $2h_{2,1} + 2$  harmonic three-forms on  $M$ .

<sup>4</sup>The Bianchi identity  $dH^{(3)} = 0$  becomes  $d\star_4 d\psi = \square_4 \psi = 0$ , which is the equation of motion for a massless  $4d$  scalar.

constant  $g_s = e^\Phi$  always belongs to this hypermultiplet.

To figure out the remaining multiplets, choose an integral basis of harmonic three-forms on the Calabi-Yau,  $(\alpha_\Lambda, \beta^\Lambda)$ , with  $\Lambda \in \{0, \dots, h_{2,1}\}$ , which satisfy

$$\int_M \alpha_\Lambda \wedge \beta^\Sigma = \delta_\Lambda^\Sigma, \quad \int_M \alpha \wedge \alpha = \int_M \beta \wedge \beta = 0,$$

define the periods as integrals of the holomorphic three-form over the 3-cycles dual to  $\alpha_\Lambda, \beta^\Lambda$  as

$$X^\Lambda = \int_{A_\Lambda} \Omega, \quad F_\Lambda = \int_{B^\Lambda} \Omega, \quad (1)$$

and let  $\omega^A$  denote an integral basis of harmonic two-forms,  $A \in 1, \dots, h_{1,1}$ . Using Greek letters for  $4d$  space-time indices and Latin letters for Calabi-Yau indices, and  $x$  for a  $4d$  spacetime coordinate, the decomposition of the various fields is then

- $G_{MN} \rightarrow g_{\mu\nu}(x)$  - the  $4d$  graviton;  $g_{m\bar{n}} = i\phi_A(x)\omega_{m\bar{n}}^A - h_{1,1}$  real  $4d$  scalars corresponding to Kähler deformations of the metric on  $M$ ;  $h_{2,1}$  complex scalars  $G^I$ ,  $I = 1, \dots, h_{2,1}$  corresponding to deformations of the complex structure of  $M^5$ . The periods  $X^\Lambda$  provide projective coordinates on the moduli space of complex structure deformations.
- $B_{MN} \rightarrow \psi(x)$  - scalar in the universal hypermultiplet, as discussed;  $B_{m\bar{n}} = b_A(x)\omega_{m\bar{n}}^A - h_{1,1}$  real scalars.
- $C_{MN}^{(2)} \rightarrow$  Similar to the  $B_{MN}$  case, we get one scalar field  $\chi(x)$  from the space-time part of  $C^{(2)}$  and another  $h_{1,1}$  real scalars  $c_A(x)$  from the two forms on  $M$ .
- $C_{MNPQ}^{(4)} \rightarrow$  Decomposing  $C^{(4)}$  as  $A^\Lambda(x)\alpha_\Lambda + \tilde{A}_\Lambda(x)\beta^\Lambda$  and imposing the self-duality condition  $F^{(5)} = \star F^{(5)}$  we get  $h_{2,1} + 1$   $4d$

<sup>5</sup>A complex structure deformation mixes the holomorphic and antiholomorphic coordinates as  $z^i \rightarrow z^i + \mu^{Ii}_{\bar{j}} z^{\bar{j}}$ . If we lower the index on  $\mu^{Ii}_{\bar{j}}$  by contracting with the holomorphic three-form  $\Omega$  the resulting  $G^I_{kl\bar{j}} = \Omega_{kli} \mu^{Ii}_{\bar{j}}$  are in one-to-one correspondence with the harmonic (2,1) forms on the Calabi-Yau.

massless  $U(1)$  gauge bosons. We also get  $h_{2,2} = h_{1,1}$  scalars from decomposing  $C^{(4)} = \gamma^A(x)\omega_A$ , where  $\omega_A$  are harmonic  $(2,2)$  forms on  $M$ .

The supersymmetry transformations (which we don't reproduce here) tell us how these bosonic fields group into multiplets. Out of the total of  $h_{2,1} + 1$  gauge fields, one linear combination - determined from the supersymmetry transformations - has to end up in the gravity multiplet and hence is called the graviphoton. We denote the graviphoton by  $A_\mu$  and the remaining gauge bosons by  $A_\mu^I$ , where  $I$  takes only  $h_{2,1}$  values. The groupings of the bosonic fields are then

- the gravity multiplet  $(g_{\mu\nu}, A_\mu)$
- $h_{2,1}$  vector multiplets  $(A_\mu^I; G^I)$
- $h_{1,1}$  hypermultiplets  $(\phi_A, b_A, c_A, \gamma_A)$
- the IIB universal hypermultiplet  $(\Phi, a, \psi, \chi)$

As shown in [9], supersymmetry does not allow couplings between vector and hypermultiplets in the leading  $4d$  effective action, if the hypermultiplets are neutral. Since  $g_s = e^\Phi$  is in a neutral hypermultiplet, supersymmetry then tells us that there are no string loop corrections to the tree-level results. In particular, the metric on the moduli space  $\mathcal{M}_V$  of the vector multiplets (the complex structure moduli for type IIB compactifications) is exact at tree level, and is read off from the kinetic terms in the  $\mathcal{N} = 2$  Lagrangian.

## 2.2. IIA Calabi-Yau compactification

We now repeat the analysis of the previous section for type IIA on a Calabi-Yau  $M$ . The  $10d$  field content is now given by the graviton  $G_{MN}$ , the NS-NS two form  $B^{(2)}$ , the dilaton  $\Phi$ , an RR two-form field strength  $F^{(2)} = dC^{(1)}$  and a four-form  $F^{(4)} = dC^{(3)}$ . The dimensional reduction of the metric gives us, as usual, the  $4d$  graviton,  $h_{1,1}$  real Kähler moduli  $\phi_A$  and  $h_{2,1}$  (complex) complex structure moduli  $G^I$ . The reduction of the  $B$ -field gives  $h_{1,1} + 1$  scalars  $b_A$  and  $\psi$ .  $C^{(1)}$  will just give one vector field and  $C^{(3)}$  will give rise to  $h_{1,1}$  vectors  $C_\mu^A$  and  $2h_{2,1} + 2$  real scalars via the decomposition  $C^{(3)} =$

$(\varphi_1 + i\varphi_2)\Omega + (\varphi_1 - i\varphi_2)\bar{\Omega} + \varphi_1^I\alpha_I + \varphi_2^I\beta_I$ . One linear combination of the vector fields will be again the graviphoton. The multiplets we obtain are then

- the gravity multiplet  $(g_{\mu\nu}, A_\mu)$
- $h_{1,1}$  vector multiplets  $(C_\mu^A, b^A, \phi^A)$
- $h_{2,1}$  hypermultiplets  $(G^I, \varphi_{1,2}^I)$
- IIA universal hypermultiplet  $(\Phi, \psi, \varphi_{1,2})$

While  $h_{1,1} \geq 1$  (the Calabi-Yau must have a  $(1,1)$  Kähler form),  $h_{2,1}$  can be zero (that is a Calabi-Yau with no complex structure deformations). In that case there is only one hypermultiplet - the universal one. Note that the numbers of vector and hypermultiplets we get in type IIA/IIB compactifications are consistent with mirror symmetry, which exchanges complex structure and Kähler moduli.

The dilaton is again in a hypermultiplet, so once more the moduli space of the vector multiplets does not get corrected by string loops<sup>6</sup>.

## 2.3. $\mathcal{N} = 2$ 4d supergravity and special geometry

$\mathcal{N} = 2$  supersymmetry highly constrains the form of the Lagrangian. As far as the scalars are concerned, we already mentioned that the moduli space of the theory factorizes into a target space  $\mathcal{M}_V$  parameterized by the vevs of the vector moduli, and  $\mathcal{M}_H$ , parameterized by the hypermultiplet scalar vevs. Supersymmetry requires that  $\mathcal{M}_V$  be a special Kähler manifold [9–11], while  $\mathcal{M}_H$  is restricted to be a quaternionic Kähler manifold [12]. In these lectures we will only be concerned with the action for the vector multiplets. The kinetic terms are determined entirely from the holomorphic prepotential  $F$  of the  $\mathcal{N} = 2$  theory, which in our context is determined by the Calabi-Yau geometry and computed from tree level string theory. A more detailed recent

<sup>6</sup>Since in type IIA compactifications the vector multiplet scalars correspond to Kähler moduli, the metric on the moduli space can receive worldsheet  $\alpha'$  corrections from string instantons wrapping the two-cycles in the Calabi-Yau (see also section 3.2). This is to be contrasted with type IIB, where the tree-level metric is exact.

review can be found in [13], whose conventions we follow.

Let us now collect a few soon-to-be-needed facts about the leading terms in the  $\mathcal{N} = 2$  Lagrangian[9,13,14], which correspond to considering just the tree level term in the Calabi-Yau prepotential ( $F = F_0$  in (48); see de Wit's lectures for more details). To be specific, we assume here that our  $\mathcal{N} = 2$  action was obtained by compactification of type IIB. In the basis introduced in the previous section, the holomorphic three-form can be written as

$$\Omega = X^\Lambda \alpha_\Lambda - F_\Lambda \beta^\Lambda. \quad (2)$$

$X^\Lambda$  (or  $F_\Lambda$ ) turn out to be projective coordinates on the vector multiplet moduli space (parameterized by  $z^i$ ,  $i \in \{1, \dots, N_V = h_{2,1}\}$ ), whose geometry is completely determined by the choice of holomorphic three-form  $\Omega(z^i)$ . The Kähler potential on  $\mathcal{M}_V$  is given by

$$\mathcal{K}(z^i, \bar{z}^i) = -\ln i \int_M \Omega \wedge \bar{\Omega}, \quad (3)$$

which can be rewritten as

$$e^{-\mathcal{K}} = i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda). \quad (4)$$

The periods  $F_\Lambda$  can be obtained from the prepotential as

$$F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda}. \quad (5)$$

One choice of (non-projective) holomorphic coordinates on  $\mathcal{M}_V$  are *special* coordinates

$$Z^I = \frac{X^I}{X^0}, \quad I \in \{1, \dots, h_{2,1}\}. \quad (6)$$

While using these coordinates is useful for a large number of purposes, please note that the symplectic invariance of the action is no longer manifest, which may sometimes not be very convenient. Note also that while  $X^\Lambda$  and  $F_\Lambda$  are holomorphic, sometimes (especially in the supergravity literature) people find it useful to define the rescaled periods  $(X'^\Lambda, F'_\Lambda) = e^{\mathcal{K}/2}(X^\Lambda, F_\Lambda)$ , which are no longer holomorphic.

Another piece of information we need from the  $\mathcal{N} = 2$  Lagrangian are the gauge kinetic terms. They read

$$\mathcal{L}_{gauge} = \frac{i}{4} \bar{\mathcal{N}}_{\Lambda\Sigma} \hat{F}_{\mu\nu}^{-\Lambda} \hat{F}^{-\Sigma\mu\nu} - \frac{i}{4} \mathcal{N}_{\Lambda\Sigma} \hat{F}_{\mu\nu}^{+\Lambda} \hat{F}^{+\Sigma\mu\nu}, \quad (7)$$

where  $\hat{F}^\pm = \frac{1}{2}(\hat{F} \mp i \star \hat{F})$  are the self-dual and anti-self-dual parts of the gauge fields<sup>7</sup>. The expression for  $\mathcal{N}_{\Lambda\Sigma}$  in terms of the prepotential and its derivatives is

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + i \frac{N_{\Lambda\Delta} X^\Delta N_{\Sigma\Omega} X^\Omega}{N_{\Delta\Xi} X^\Delta X^\Xi}, \quad (8)$$

where  $F_{\Lambda\Sigma}$  denote the second derivatives of the prepotential with respect to  $X^\Lambda$  and  $X^\Sigma$  and  $N_{\Lambda\Sigma} = -i(F_{\Lambda\Sigma} - \bar{F}_{\Lambda\Sigma})$ . One can easily show that

$$F_\Lambda(X) = \mathcal{N}_{\Lambda\Sigma} X^\Sigma \quad (9)$$

a relation that will be useful later.

The integral of  $\hat{F}$  over a sphere at infinity gives the magnetic charge of the field configuration we are studying.

$$p^\Lambda = \frac{1}{4\pi} \int_{S^2} \hat{F}^\Lambda = \frac{1}{2\pi} Re \int_{S^2} \hat{F}^{+\Lambda}. \quad (10)$$

The total electric charge is usually given by the integral of the dual field strength over the sphere at infinity, which follows from the Maxwell equation  $d \star F = 0$ . Note that our action (2.3) gives rise to different equations of motion, which imply that the electric charge is given by

$$q_\Lambda = \frac{1}{2\pi} Re \int_{S^2} G_\Lambda^+ \quad (11)$$

where

$$G_\Lambda^+ = \mathcal{N}_{\Lambda\Sigma} \hat{F}^{\Sigma+} \quad (12)$$

or equivalently

$$G_\Lambda = (Re \mathcal{N}_{\Lambda\Sigma}) \hat{F}^\Sigma + (Im \mathcal{N}_{\Lambda\Sigma}) \star \hat{F}^\Sigma \quad (13)$$

<sup>7</sup>In four Lorentzian dimensions, (anti)self-duality reads  $\star \hat{F}^\pm = \pm i \hat{F}^\pm$ .

One last thing we would like to explain is how to recognise which linear combination of the gauge fields is the graviphoton. We know that the particular combination has to be symplectically invariant, since which field is in the gravity multiplet should not depend on the choice of symplectic basis.

From the field strengths  $\hat{F}^\Lambda, G_\Lambda$  and the periods one can construct a naturally symplectically-invariant field strength, given by

$$T_{\mu\nu}^- = F_\Lambda \hat{F}_{\mu\nu}^{-\Lambda} - X^\Lambda G_\Lambda^-, \quad (14)$$

which corresponds to the graviphoton. The graviphoton is also special in that its charge

$$Q = \frac{1}{4\pi} \oint T^- \quad (15)$$

is proportional to the central charge of the  $\mathcal{N} = 2$  supersymmetry algebra<sup>8</sup>. If we restrict our attention to just supersymmetric solutions (preserving 4 or 8 supercharges), then

$$F_\Lambda \hat{F}^{\Lambda} - X^\Lambda G_\Lambda^+ = 0 \quad (16)$$

when evaluated on these solutions, so the graviphoton charge becomes

$$Q = F_\Lambda p^\Lambda - X^\Lambda q_\Lambda = Q_{mag} + iQ_{el}, \quad (17)$$

which generally is complex.

Having talked at length about electric and magnetic charges in the  $4d$  theory, we might as well mention how to obtain charged particles in string theory compactified on a Calabi-Yau  $M$ . The answer is simple: just wrap D-branes on the various cycles in  $M$ . The string theory D-branes source the  $10d$  RR fields, which from the  $4d$  point of view look like pointlike charges that source the different gauge fields which come from the dimensional reduction of the  $10d$  RR fields. In type IIB, D3-branes can wrap any of the  $2h_{2,1} + 2$  different 3-cycles, giving a total of  $2h_{2,1} + 2$  different electric and magnetic charges in  $4d$ . The number of units of  $4d$  charge is determined by how many times we wrap the  $D$ -branes around the particular cycle.

<sup>8</sup>Note that the  $Q$  so defined rescales under projective transformations.

In type IIA the stable D-branes are even-dimensional. One can again produce any electric and magnetic charges by wrapping D6, D4, D2 and D0 branes on the various cycles in the Calabi-Yau. If  $A \in \{1, \dots, h_{1,1}\}$  labels the 2 (and also the dual 4)-cycles, then the most general set of charges we can get is  $(p^0, p^A, q_A, q_0)$ , which stands for D6, D4 (magnetic) and D2, D0 (electric) charges respectively.

If we wrap a large number of branes at the same point in noncompact  $4d$  space, we will have to consider the backreaction of the metric and the other supergravity fields. It turns out that for large charges one can obtain macroscopic black holes [15,16], which we will now turn to study.

## 2.4. Black hole solutions

In a classic paper [17], Gibbons and Hull have shown that minimal  $4d$   $\mathcal{N} = 2$  supergravity (whose bosonic sector is just Einstein - Maxwell gravity) has charged black hole solutions of the Reissner-Nordström type. These solutions are parametrised by their mass  $M$  and charge  $Q = Q_{mag} + iQ_{el}$ . Since  $Q$  happens to equal the central charge of the  $\mathcal{N} = 2$  supersymmetric theory, then the BPS bound requires that these black holes have  $M \geq |Q|$ . If  $M > |Q|$  their Hawking temperature is nonzero, which means they can radiate and thus are not stable objects. Since we are looking for supersymmetric solutions, which have to be stable, the remaining candidate is the extremal solution with  $M = |Q|$ . The Hawking temperature of these black holes is zero and the solution indeed turns out to be supersymmetric. These extremal objects have been the focus of many interesting investigations over the years.

The easiest way to see whether a particular solution is supersymmetric is to look at the fermion variations for that particular background and require that they vanish. The only fermions present in minimal supergravity are the gravitini, whose variation is

$$\delta_\epsilon \psi_\mu^\alpha = 2\nabla_\mu \epsilon^\alpha - \frac{i}{8} T_{\lambda\nu}^- \gamma^{\lambda\nu} \gamma_\mu \epsilon^{\alpha\beta} \epsilon_\beta. \quad (18)$$

The solution will be supersymmetric if there exists a spinor  $\epsilon^\alpha$  such that  $\delta_\epsilon \psi_\mu = 0$ . It turns out that there exist four such spinors, so the solution

preserves half of the original eight supersymmetries.

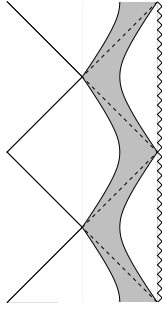


Figure 2. Penrose diagram for the extremal Reissner-Nordström black hole. The zigzag stands for the timelike singularity (at  $r = -Q$ ) and the dotted lines represent event horizons. The shaded region covers the near-horizon  $AdS_2 \times S^2$ , illustrated in figure 3.

The expression for the metric is

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} (dr^2 + r^2 d\Omega_2^2), \quad (19)$$

where

$$e^{-U(r)} = 1 + \frac{|Q|}{r}. \quad (20)$$

The solution carries charge  $Q$ , as the expression we get for the graviphoton indicates

$$T_{\mu\nu}^- = Q \epsilon_{\mu\nu}^-, \text{ with } \int_{S^2} \epsilon^- = 4\pi. \quad (21)$$

As  $r \rightarrow \infty$ ,  $e^U \rightarrow 1$  and the metric becomes just the flat metric on  $\mathbb{R}^{3,1}$ . As  $r \rightarrow 0$  the metric takes the form

$$ds^2 = -\frac{r^2}{|Q|^2} dt^2 + \frac{|Q|^2}{r^2} dr^2 + |Q|^2 d\Omega_2^2. \quad (22)$$

The near-horizon geometry is thus  $AdS_2 \times S^2$ , and the area of the horizon is  $\mathcal{A} = 4\pi|Q|^2$ . The near horizon bosonic isometry group is  $SO(2,1) \times SU(2)$ , which is part of the  $SU(1,1|2)$  superisometry group containing the maximal eight supersymmetries. This means that  $AdS_2 \times S^2$  is a maximal  $\mathcal{N} = 2$  vacuum, and hence the Reissner-Nordström solution we obtained can be thought of as a soliton radially interpolating between two maximally supersymmetric vacua [18].

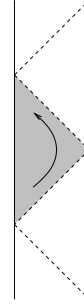


Figure 3. Penrose diagram for  $AdS_2$ . The dotted lines represent the horizons inherited from the embedding in the extremal Reissner-Nordström geometry of figure 2.

## 2.5. The attractor equations

Next, we add in vector multiplets [19,20]. The gravitini variations now acquire extra terms, which contain derivatives of the vector moduli

$$\delta_\epsilon \psi_\mu^\alpha = 2\nabla_\mu \epsilon - \frac{i}{8} T_{\nu\lambda}^- \gamma^{\nu\lambda} \gamma_\mu \epsilon^{\alpha\beta} \epsilon_\beta + i A_\mu \epsilon^\alpha, \quad (23)$$

where

$$A_\mu = \frac{i}{2} N_{\Lambda\Sigma} (\bar{X}^\Lambda \partial_\mu X^\Sigma - \partial_\mu \bar{X}^\Lambda X^\Sigma) \quad (24)$$

where in the above two and the following equations we have fixed the gauge  $N_{\Lambda\Sigma} X^\Lambda \bar{X}^\Sigma = -1$ .



The supersymmetry variations of the fermions in the vector multiplets (the gaugini) are

$$\begin{aligned}\delta\Omega_\alpha^\Lambda &= 2\gamma^\mu(\partial_\mu + iA_\mu)X^\Lambda\epsilon_\alpha \\ &+ \frac{1}{2}\gamma^{\nu\lambda}(\hat{F}_{\nu\lambda}^{+\Lambda} - \frac{1}{4}X^\Lambda T_{\nu\lambda}^+)\epsilon_{\alpha\beta}\epsilon^\beta\end{aligned}\quad (25)$$

We would like to see if a maximally supersymmetric near-horizon  $AdS_2 \times S^2$  region with an  $SU(1,1|2)$  superisometry group can still exist. This requires that the moduli  $X^\Lambda$  be constant throughout the near-horizon spacetime. Then the gravitini variations vanish as before and the gaugino equations require that

$$\hat{F}_{\mu\nu}^{+\Lambda} = \frac{1}{4}X^\Lambda T_{\mu\nu}^+. \quad (26)$$

Integrating our previous solution for the graviphoton  $T_{\mu\nu}^+ \propto \epsilon_{\mu\nu}^+$  (where  $\epsilon^+$  is a self-dual two-form) over the horizon  $S^2$  we find

$$p^\Lambda = \frac{1}{2\pi} \int_{S^2} Re \hat{F}^{+\Lambda} = Re[CX^\Lambda], \quad (27)$$

for some complex constant  $C$ , which in general will depend on our choice of projective gauge<sup>9</sup>. Similarly,

$$\begin{aligned}q_\Lambda &= \frac{1}{2\pi} \int_{S^2} Re \hat{G}_\Lambda^+ = \frac{1}{2\pi} \int_{S^2} Re N_{\Lambda\Sigma} \hat{F}^{+\Sigma} \\ &= Re[CN_{\Lambda\Sigma}X^\Sigma] = Re[CF_\Lambda].\end{aligned}\quad (28)$$

We thus see that the moduli  $X^\Lambda$ ,  $F_\Lambda$  must take very specific values in terms of the black hole charges in order to recover the  $AdS_2 \times S^2$  part of the solution in the near-horizon region. For reasons which will become clear shortly these equations are known as the *attractor equations*.

The attractor equations, together with knowledge of the prepotential  $F(X^\Lambda)$ , generically let us determine in principle (up to possible discrete choices) the real and imaginary parts of all  $CX^\Lambda|_{hor}$  in terms of the charges of the black hole. This can be easily checked by counting the number of equations and unknowns.

The Bekenstein-Hawking entropy depends only on the near-horizon data and is simply given by the horizon value of

$$S_{BH} = \frac{\pi i}{2}(q_\Lambda \bar{C} \bar{X}^\Lambda - p^\Lambda \bar{C} \bar{F}_\Lambda). \quad (29)$$

<sup>9</sup>For  $N_{\Lambda\Sigma}X^\Lambda\bar{X}^\Sigma = -1$ , we have  $\bar{C} = 2iQ$ .

After solving the attractor equations, the entropy becomes just a function of the charges of the black hole, and does not depend on any asymptotic data, such as the asymptotic values of the moduli (as long as we do not leave the basin of attraction).

In general one might want to find the full black hole solution, including the asymptotic region. This is a harder problem, and one we will not need to solve for now because the full solution has the form of a radially interpolating soliton, with the moduli in the maximally-symmetric near horizon region determined by the charges of the black hole via the attractor equations. So, given the charges and the value  $X_\infty^\Lambda$  of the moduli at infinity, the variation through space of  $X^\Lambda$  will be such that it always goes to the horizon value  $X_{hor}^\Lambda$  determined by the attractor equations [21]. The path traced in moduli space by  $X^\Lambda(r)$ <sup>10</sup> is called an *attractor flow*.

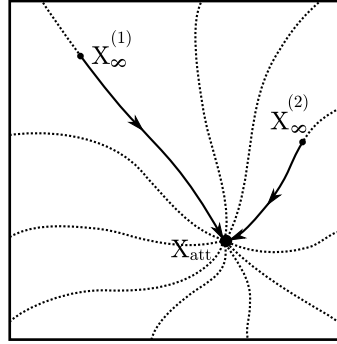


Figure 4. Attractor flow in moduli space. No matter which point (within the basin of attraction) in the moduli space at  $r = \infty$  we choose as our asymptotic data, at the horizon ( $r = 0$ ) the moduli always take the same attractor value, determined by the charges of the black hole.

While at first this behavior of the vector moduli might seem a bit surprising, it is actually re-

<sup>10</sup>We here only consider spherically-symmetric solutions. It turns out there also exist multicenter solutions [1].

quired for the existence of sensible counting of microstates of the black hole: the area of the horizon depends on  $X_{hor}^\Lambda$ , but at the same time it is the logarithm of the number of microstates of the black hole - that is - the log of an integer. Therefore one does not expect it to smoothly depend on continuous parameters such as the asymptotic values of the moduli.<sup>11</sup> Additionally, the entropy is not expected to depend on parameters that we may choose to tune at infinity, since it is an intrinsic property of the black hole. Note that there is no attractor mechanism for the hypermultiplet moduli, but this does not affect the above reasoning, since the horizon hyperscalar vevs drop out of the entropy formula<sup>12</sup>.

### 2.5.1. A simple example

Maybe the simplest example [30] is to consider IIA compactified on a Calabi-Yau with moduli  $X^\Lambda = (X^0, X^A)$ . In the large-volume limit we can take the Calabi-Yau prepotential to be

$$F(X) = D_{ABC} \frac{X^A X^B X^C}{X^0}, \quad (30)$$

where  $6D_{ABC}$  are the intersection numbers of the Calabi-Yau. In that case we have

$$\begin{aligned} F_A &= \partial_A F = \frac{3D_{ABC} X^B X^C}{X^0} \\ F_0 &= \partial_0 F = -\frac{D_{ABC} X^A X^B X^C}{(X^0)^2}. \end{aligned} \quad (31)$$

We will consider a  $D4$ - $D0$  black hole, with charges  $p^0 = q_A = 0$  and  $p^A, q_0 \neq 0$ . The attractor equations read

$$\begin{aligned} \text{Re}[CX^A] &= p^A, & \text{Re}[CX^0] &= 0, \\ \text{Re}[CF_A] &= 0, & \text{Re}[CF_0] &= q_0. \end{aligned} \quad (32)$$

<sup>11</sup>However a more refined analysis shows that it can depend *discontinuously* on the asymptotic moduli[1]. This leads to very interesting modifications of the simplified picture presented here. See [22] for a recent discussion.

<sup>12</sup>In recent work by Sen [23], it was shown that all fields that the entropy depends on at the horizon must be subject to an attractor mechanism. His reasoning works not only for supersymmetric black holes, but any black hole whose near-horizon symmetry group contains  $SO(2,1)$  [24–29].

It is easy to see that the solution to the above equations is

$$CX^A = p^A, \quad CX^0 = i\sqrt{\frac{D}{q_0}}, \quad (33)$$

where we have defined  $D \equiv D_{ABC} p^A p^B p^C$ . From (29) we can compute the macroscopic entropy associated with this black hole

$$S_{BH} = \frac{\pi i}{2} (q_0 \bar{C} \bar{X}^0 - p^A \bar{C} \bar{F}_A) = 2\pi \sqrt{D q_0}. \quad (34)$$

## 2.6. The general asymptotically flat solution

Somewhat surprisingly, given a solution of the near-horizon attractor equations for the vector moduli as a function of the charges, it is possible to actually construct the full asymptotically flat solution with generic values for the moduli at infinity. Here we will concentrate only on the spherically symmetric case, although generic exact multi-center solutions are also known [1].

Due to spherical symmetry, the metric must take the form (19). The insight of [31–33], was to notice that the moduli fields that satisfy the BPS equations of motion [34] must obey equations that are very similar to the attractor equations (27) and (28), just that they hold throughout spacetime, and not only at the horizon of the black hole

$$\begin{aligned} \text{Re}[C(r)X^\Lambda(r)] &= H^\Lambda(r) \\ \text{Re}[C(r)F_\Lambda(X(r))] &= H_\Lambda(r). \end{aligned} \quad (35)$$

Here  $H^\Lambda, H_\Lambda$  are harmonic functions on flat  $\mathbb{R}^3$  sourced by the charges  $p^\Lambda, q_\Lambda$ . It follows that whenever one can solve the attractor equations (which is generally a difficult task), one can also get the solution for the moduli everywhere<sup>13</sup>. This is done with the aid of the *entropy function*,  $\Sigma$ , defined as

$$\Sigma(\vec{x}) = \frac{1}{2} \text{Im}[CX^\Lambda(\vec{x}) \bar{C} \bar{F}_\Lambda(\vec{x})]. \quad (36)$$

<sup>13</sup>When the attractor equations are not analytically solvable, spacetime solutions can still be explicitly written in terms of the implicit attractor solutions.

By comparison with (29), we see that near the horizon  $\Sigma(r, p^\Lambda, q_\Lambda) = (\pi r^2)^{-1} S_{BH}(p^\Lambda, q_\Lambda)$ . Away from the horizon,  $\Sigma(r)$  is simply given by making the replacement

$$\begin{aligned} p^\Lambda &\rightarrow H^\Lambda(r) = \frac{p^\Lambda}{r} + h^\Lambda \\ q_\Lambda &\rightarrow H_\Lambda(r) = \frac{q_\Lambda}{r} + h_\Lambda \end{aligned} \quad (37)$$

in the entropy formula  $\frac{1}{\pi} S_{BH}(p^\Lambda, q_\Lambda)$ . The constants  $h^\Lambda, h_\Lambda$  can be determined from the asymptotic values of the moduli. Finally, the solution for the metric and the moduli is

$$e^{-2U} = \Sigma(H), \quad CX^\Lambda = H^\Lambda + i \frac{\partial \Sigma(H)}{\partial H_\Lambda}. \quad (38)$$

This is a very powerful result, since it allows us to reconstruct the solution for the metric and moduli *throughout spacetime* from just knowledge of the entropy as a function of the charges and the asymptotic values of the moduli.

There is a catch though, in that even if the entropy itself is real, the entropy function is not guaranteed to be so.  $\Sigma^2(r)$  is a quartic polynomial in  $\frac{1}{r}$ , which is positive as  $r \rightarrow 0, \infty$ , but there is nothing to prevent it from becoming negative for an intermediate range of  $r$ , if we tune the moduli at infinity appropriately. Since  $\Sigma(r)$  is a metric component, the solution becomes unphysical if it becomes imaginary. Therefore one must always check whether our formal solution is actually physical by making sure that  $\Sigma(r)$  is real and the moduli  $\tau^A(r)$  belong to the physical moduli space for all values of the radius.

Even if the asymptotic moduli are outside the regions for which the single-centred solution exists, the respective BPS state might still be realized as a black hole (or point particle) bound state [35]. The entropy function can be adapted [1] to describe these multicenter supersymmetric black holes. One simply allows the harmonic functions to have poles at the location of each black hole carrying the corresponding charge. These solutions have many interesting properties and applications [4, 35–39, 22], but are outside the scope of these lectures.

Note that checking the reality of  $\Sigma(\vec{x})$  in the case of multicenter solutions is an extremely dif-

ficult task even for simple Calabi-Yaus, which could only be tackled numerically.

### 2.6.1. Simple example redux

As an example, let us again take the D4-D0 black hole, but now in a Calabi-Yau compactification that has only one complex Kähler modulus  $\tau(\vec{x})$ , where  $\vec{x}$  denotes position in  $\mathbb{R}^3$ , and the triple self-intersection number of the Calabi-Yau is  $D_{111} = 1$ . The entropy is then  $S = 2\pi\sqrt{q_0 p^3}$ . The entropy function is

$$\Sigma(H(\vec{x}), H_0(\vec{x})) = 2\sqrt{H_0(\vec{x})H^3(\vec{x})} = e^{-2U(\vec{x})}, \quad (39)$$

where the harmonic functions are

$$H = \frac{p}{r} + h, \quad H_0 = \frac{q_0}{r} + h_0. \quad (40)$$

The solution for the modulus in this case is purely imaginary

$$\tau(\vec{x}) = \frac{CX(\vec{x})}{CX^0(\vec{x})} = -i\sqrt{\frac{H_0(\vec{x})}{H(\vec{x})}} \quad (41)$$

so now we only need to determine the constants  $h_0, h$  in terms of the asymptotic value of the modulus  $\tau_\infty = -ia_\infty$  (here the physical moduli space is the lower half plane). We choose a gauge so that as  $r \rightarrow \infty$  the scale factor  $e^{-2U} \rightarrow 1$ , which imposes the constraint  $2\sqrt{h_0 h^3} = 1$ . We then get  $h_0 = 2^{-\frac{1}{2}}(i\tau_\infty)^{\frac{3}{2}}$ ,  $h = (2i\tau_\infty)^{-\frac{1}{2}}$ . Note that the asymptotic Kähler class  $i\tau_\infty$  has to be positive for the solution to exist.

The attractor equations only require that the B-field (proportional to  $Re\tau$ ) at the horizon be zero, but its value at infinity does not have to vanish, as our solution seems to indicate. Nonzero  $B_\infty$  can be obtained by considering the entropy function associated to a black hole with additional  $D2$  charges, but in which we take the corresponding  $H_A = \text{constant}$ .

### 2.6.2. Generic entropy for cubic prepotential

We now give the formula [30] for the entropy of the generic D6-D4-D2-D0 black hole when the prepotential is simply

$$F = \frac{D_{ABC}X^A X^B X^C}{X^0}, \quad (42)$$

Then the entropy of the black hole with charges  $(p^0, p^A, q_A, q_0)$  is

$$S = 2\pi\sqrt{Q^3 p^0 - J^2 (p^0)^2} \quad (43)$$

where  $Q$  is determined by solving the following equations for a set of variables  $y^A$

$$3D_{ABC}y^A y^B = q_A + \frac{3D_{ABC}p^B p^C}{p^0} \quad (44)$$

which for general charges and intersection numbers are not analytically soluble. In any case, we have

$$Q^{\frac{3}{2}} = D_{ABC}y^A y^B y^C \quad (45)$$

and

$$J = -\frac{q_0}{2} + \frac{D_{ABC}p^A p^B p^C}{(p^0)^2} + \frac{p^A q_A}{2p^0}. \quad (46)$$

Please note that our analysis is only valid when the Calabi-Yau is large both at infinity and at the horizon, since the entropy formula was derived using the large volume prepotential (30). One has to pay special attention if the attractor flow in question passes through a region in moduli space where the Calabi-Yau (or some cycle in the Calabi-Yau) is becoming small, since then instanton corrections to the prepotential become important [22]. In order for our supergravity analysis to be valid, we also need the  $4d$  curvature to be small everywhere, which translates into the requirement of large black hole charges.

## 2.7. Higher orders

### 2.7.1. Corrections to the $\mathcal{N} = 2$ action

In this subsection we briefly sketch how the  $\mathcal{N} = 2$  invariant Lagrangian, including higher curvature corrections, is constructed using the so-called superconformal tensor calculus [9]. The idea is to begin by constructing an action for  $\mathcal{N} = 2$  conformal supergravity coupled to  $N_V + 1$  conformal vector multiplets and then gauge-fix it down to Poincaré supergravity coupled to  $N_V$  vector multiplets. The advantage of this approach relies on the fact that more symmetries

are realized in a linear and simple way in conformal supergravity, and also the off-shell multiplets are smaller<sup>14</sup>. The basic ingredients are two types of superconformal multiplets

- the Weyl multiplet, which contains the vierbein  $e_\mu^a$  and the two gravitini, among many other auxiliary fields. To incorporate it into the action, one actually has to construct an  $\mathcal{N} = 2$  chiral multiplet  $W^2$ , which contains the gauge-invariant field strengths<sup>15</sup>
- $N_V + 1$  vector multiplets, each containing a scalar  $X^\Lambda$ , a gaugino  $\Omega^\Lambda$  and a gauge field  $A_\mu^\Lambda$ , among other stuff. One of these vector multiplets will provide the graviphoton for the Poincaré gravity multiplet, since the Weyl multiplet does not contain an independent gauge field.

The action for the vector multiplets  $\mathbf{X}^\Lambda$  (where boldface type denotes a superfield) is constructed by taking a holomorphic function  $F(\mathbf{X}^\Lambda)$  and integrating it over  $\mathcal{N} = 2$  superspace. Dilation invariance requires that  $F$  be homogenous of degree two in the  $\mathbf{X}^\Lambda$ . To recover minimal supergravity we should only consider one vector multiplet  $\mathbf{X}$ , for which the prepotential is  $F(\mathbf{X}) = \mathbf{X}^2$ . Integrating over superspace we obtain

$$\int d^4\theta F(\mathbf{X}) = X(\Box_4 - \frac{1}{6}R)\bar{X} + \dots \quad (47)$$

Upon conformal gauge-fixing  $X = \text{const}$  this gives rise to the Einstein-Hilbert term. The Maxwell term comes about in the usual way [40]. A similar mechanism will give rise to the Einstein-Hilbert, Maxwell and scalar terms in the non-minimal action. Note that we will get only  $N_V$  scalars out of the initial  $N_V + 1$ , since one combination gets gauge-fixed to a constant.

In order to include higher curvature terms in the Lagrangian, one has to add couplings to the

<sup>14</sup>Indeed the simplifications are so striking one suspects there may be some deeper physical significance underlying this "mathematical trick".

<sup>15</sup>Very roughly, the highest component of  $W^2$  is the anti-selfdual part of the Weyl tensor squared, while the lowest component is the square of the anti-selfdual part  $(T^-)^2$  of an auxiliary tensor field  $T$  that gets identified with the graviphoton upon gauge-fixing.

chiral multiplet  $W^2$ . This is achieved by simply extending the holomorphic prepotential to also be a function of  $W^2$ ,  $F(\mathbf{X}^\Lambda, W^2)$ , which can be expanded as

$$F(\mathbf{X}^\Lambda, W^2) = \sum_{g=0}^{\infty} F_g(\mathbf{X}^\Lambda) W^{2g} \quad (48)$$

The  $F_g$  are now required to be homogenous of degree  $2-2g$ . Quite interestingly, they are related to topological string genus  $g$  amplitudes, as we will explore later in these notes.

### 2.7.2. Wald's formula

The Bekenstein-Hawking area law for the macroscopic entropy of a black hole was derived in Einstein gravity. It cannot possibly remain exactly valid when higher curvature corrections to the action are included, as the area is not invariant under field redefinitions (e.g. which mix  $g_{\mu\nu}$  and  $R_{\mu\nu}$ ), while the entropy must be. It was shown in [41,42] how the area law has to be modified in the presence of  $R^2$  or higher derivative terms in the Einstein action in order that the first law of black hole mechanics - the spacetime manifestation of the first law of thermodynamics - remain valid. The first law of black hole mechanics can be put in the usual form

$$\delta M = \frac{\kappa_S}{2\pi} \delta S + \phi \delta q + \mu \delta p, \quad (49)$$

where  $\kappa_S$  is the surface gravity on the horizon, if  $S$  is given by<sup>16</sup>

$$S = 2\pi \int_{\mathcal{H}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} d\Omega. \quad (50)$$

Here  $\epsilon_{\mu\nu}$  is the binormal to the horizon  $\mathcal{H}$  and  $d\Omega$  is the volume element on  $\mathcal{H}$ . Note that if the Lagrangian only consists of the Einstein-Hilbert term, then  $S$  equals the area of the horizon, but  $R^2$  and higher curvature corrections to the action do generically modify the area law.

Thus, upon adding higher curvature terms to the  $\mathcal{N} = 2$  supergravity action, the entropy of

<sup>16</sup>This is the correct formula in the case in which the effective action contains powers of the Riemann tensor, but no derivatives thereof.

the black hole solutions gets modified in two ways: first, the metric on the horizon changes as a consequence of the modified equations of motion. Second, the entropy formula itself receives corrections according to (50).

The case that concerns us - of BPS black holes in  $\mathcal{N} = 2$  supergravity coupled to vector multiplets - was considered in [43–47]. The authors argued that after adding the terms (48) to the  $\mathcal{N} = 2$  supergravity action, the near horizon geometry was still  $AdS_2 \times S^2$ , and that the moduli were still subject to an attractor mechanism. Their horizon values were fixed by the following generalisation of the attractor equations

$$Re CX^\Lambda = p^\Lambda, \quad Re CF_\Lambda = q_\Lambda \quad (51)$$

$$C^2 W^2 = 256 \quad (52)$$

where now  $F_\Lambda$  is the derivative of the full corrected prepotential  $F(X^\Lambda, W^2)$ . Taking into account the combined effect of the metric backreaction and Wald's corrections to the area law, the expression for the entropy becomes

$$S_{BH} = \frac{\pi}{2} Im(CX^\Lambda \bar{C}\bar{F}_\Lambda) - \frac{\pi}{2} Im(C^2 W \partial_W F) \quad (53)$$

evaluated on the horizon. In lecture 3 we will use this result to match the perturbative expansion of the black hole partition function to the perturbative expansion of the topological string.

### 2.8. The MSW CFT

In this subsection we discuss the CFT duals known for a large class of Calabi-Yau black holes. Finding the dual for the most general case is a very interesting unsolved problem. A dual CFT description [48] (often known as the MSW CFT) has been proposed in the still fairly general case in which the D6 charge is zero, but the D4-D2-D0 charges and the Calabi-Yau itself are (almost) arbitrary. We will denote the charges by  $(p^A, q_A, q_0)$  and the intersection numbers of the Calabi-Yau  $M$  by  $6D_{ABC}$ . As before, we define  $D = D_{ABC} p^A p^B p^C$ .

Upon lifting to M-theory, the D4-D2-D0 brane configuration becomes an M5-brane wrapped on  $P \times S^1$ , where the surface  $P$ <sup>17</sup> is holomorphic-

<sup>17</sup>The charges must be such that  $P$  is very ample [48], which is the case if the Kähler class is large.

cally embedded in  $M$  (as a consequence of supersymmetry) and  $S^1$  stands for the M-theory circle all throughout this lecture.  $P$  decomposes as  $P = p^A \Sigma_A$  where  $\Sigma_A$  is an integral basis of 4-cycles on  $M$ . The M5 carries worldvolume fluxes that give rise to induced M2-brane charges ( $q_A$ ), as well as  $q_0$  units of momentum along the  $S^1$ . From the supergravity point of view, under the M-theory lift the D4-D2-D0 black hole becomes an  $S^1$ -wrapped black string, whose near horizon geometry is locally  $AdS_3 \times S^2 \times M$ . Note that the M-theory lift of a D6-brane is Taub-Nut space, so if we considered black holes with D6 charge the eleven-dimensional geometry would no longer be  $AdS_3$ , which is partly the reason that a microscopic description is not known in that case<sup>18</sup>.

The low energy dynamics on the M5 worldvolume is captured by an effective 2d CFT living on  $S^1 \times \mathbb{R}$  (time). This 2d CFT has  $(0, 4)$  supersymmetry, inherited via dimensional reduction from the  $(0, 2)$  supersymmetry of the 6d theory living on the M5 worldvolume. Its low-energy excitations arise as zero-modes of the fluctuations of the M5 worldvolume fields (embedding, self-dual 3-form  $h^{(3)}$  and the right-moving (RM) 6d fermions  $\psi^{(6)}$ ) on  $P$ , as follows

- zero-mode fluctuations of the embedding correspond to cohomology classes on  $P$ .
- zero-modes of the  $h$ -field correspond to self- and anti-self-dual forms on  $P$ , as can be seen from the decomposition

$$h^{(3)} = d\phi^A \wedge \alpha_A, \quad \alpha_A \in H^2(P, \mathbb{Z})$$

If the 2-form  $\alpha_A$  is self-dual, then the self-duality of  $h$  implies that the scalar modulus  $\phi^A$  has to be right-moving, while if  $\alpha^A$  is anti-self-dual, then  $\phi^A$  is left-moving. Thus we obtain  $b_2^-$  LM and  $b_2^+$  RM scalars in the CFT from the dimensional reduction of the  $h$ -field, where  $b_2^\pm$  are the numbers of self-dual and respectively anti-self-dual two-forms on  $P$ .

<sup>18</sup>Quite surprisingly, the MSW string remakes its appearance for certain black holes with nonzero D6 charge [49], which may imply that *all* IIA black holes are described by some deformation of the MSW CFT. This makes it all the more worth studying, of course.

- fermions in the CFT arise from  $(0, 2)$  forms on  $P$  and they are RM. This can be easily seen by decomposing the 6d RM fermions as

$$\psi_{(6)} = \sum_I \psi_{(2)}^I \otimes \psi_I^P$$

where  $\psi_I^P$  are fermionic zero-modes on  $P$ , which are known to be in one-to-one correspondence with harmonic  $(0, 2)$  forms.<sup>19</sup> The number of RM fermions is  $4h_{2,0}(P)$  and it can be shown to equal the number of RM bosons, as required by supersymmetry on the right.

- there is one distinguished  $\mathcal{N} = 4$  multiplet in the 2d CFT, called the *centre of mass multiplet*. Its bosonic content is given by the three massless scalars  $X^i$  that parameterize the motion of the black hole as a whole in the three noncompact directions and one right-moving mode  $\varphi$  of the  $h$  field, which corresponds to the unique self-dual form on  $P$  which is extendible to a 2-form on  $M$ . This is of course the pullback of the Kähler form on  $M - J$  - which has to be proportional to  $[P]$  at the horizon, as a consequence of the attractor equations. In terms of the scalars  $\phi^A$

$$\varphi = p^A D_{AB} \phi^B \equiv p^A \phi_A. \quad (54)$$

The fermionic partners of these four bosons are the goldstinos  $\tilde{\psi}^{\pm\pm}$  that arise from the four supersymmetries broken by the brane configuration.

The resulting central charges, including a sub-leading correction proportional to the second Chern class of  $M$ , are [48]

$$c_L = 6D + c_2 \cdot P, \quad c_R = 6D + \frac{1}{2} c_2 \cdot P. \quad (55)$$

The MSW CFT reproduces the area-entropy law. If one is only interested in the D4-D0 system, then the left-moving oscillator momentum is

<sup>19</sup>There are no left-moving fermions because they would be in one-to-one correspondence with  $(0, 1)$  forms on  $P$ , of which there are none since  $b_1(P) = b_1(M) = 0$ .

$q_0$ , while the right-moving oscillator momentum has to be zero by supersymmetry. Using Cardy's formula, the entropy reads

$$S = 2\pi\sqrt{\frac{cLq_0}{6}} = 2\pi\sqrt{Dq_0} \quad (56)$$

in agreement with the macroscopic formula (34).

What happens if we add D2/M2 charges? This corresponds to turning on M2 brane fluxes (that is, nonzero  $h^{(3)}$  flux on cycles of the form  $S^1 \times \alpha$ , with  $\alpha$  a two-cycle on  $P$ ) on the M5 worldvolume. In the effective 2d theory, membrane charge is the zero-mode momentum  $q_A = \int d\phi_A$  carried by the massless scalars that arise from the dimensional reduction of the three-form  $h$ , and thus it is a vector in the Narain lattice of massless scalars.  $q_A$  contributes to the  $S^1$  momentum along the string. The effect is to shift the momentum available to be distributed among the LM oscillators by

$$q_0 \rightarrow \hat{q}_0 = q_0 + \frac{1}{12}D^{AB}q_Aq_B, \quad (57)$$

$D^{AB}$  is the inverse of the matrix of charges  $D_{AB} = D_{ABC}p^C$ . The entropy gets modified to  $S = 2\pi\sqrt{\hat{q}_0 D}$ , and it is straightforward to check that this agrees with the supergravity formula (43) for  $p^0=0$ .

## 2.9. The (modified) elliptic genus

So far we have been loosely speaking about the ‘‘entropy’’ of the black hole and its CFT dual. In fact, what we are really looking for is a BPS protected quantity which does not change as one extrapolates between the supergravity and CFT regimes. Such an object is the (modified) elliptic genus, which is an index that counts a weighted number of ground states. At leading order (given by the area law), which quantity we use seems not to matter much<sup>20</sup>, but at subleading orders we will definitely need to be more precise.<sup>21</sup>

<sup>20</sup>Presumably what happens is that one or both of  $n_B$  and  $n_F$  are proportional to  $e^{Area/4G}$ , but with different proportionality constants, so that the difference between the log of the total number of states and the log of the index is subleading.

<sup>21</sup>Actually, the first subleading correction to the entropy is included in (43) expression and matches [48]: on the microscopic side it involved the  $c_2 \cdot P$  correction to the central charge, and on the macroscopic side it came from the Gauss-Bonnet term in the 4d effective action.

Let us give a very simple example of what an index is [50]. Take supersymmetric quantum mechanics with one supercharge  $Q$  ( $Q^2 = H$ ), and define the Witten index as

$$I_F = \text{Tr}_{\text{states}} (-1)^F e^{-\beta H} \quad (58)$$

where  $F$  is the fermion number operator. Since for states of nonzero energy  $|E\rangle$ ,  $Q|E\rangle$  is a state with the same energy but opposite fermion number (mod 2),  $I_F$  only gets contributions from the ground states, since they are annihilated by  $Q$  and thus are not paired. If we denote by  $n_B$  and  $n_F$  the number of bosonic and respectively fermionic ground states, then  $I_F = n_B - n_F$ . The index is rigid under small deformations of the parameters in the Hamiltonian.

This simple index actually vanishes for the MSW CFT [51,52], whose symmetry algebra on the right is a Wigner contraction of the large  $\mathcal{N} = 4$  superconformal algebra. This consists of a small  $\mathcal{N} = 4$  superconformal algebra plus four bosonic and four fermionic generators, which are nothing but the fields in the center-of-mass multiplet  $(X^i, \varphi, \tilde{\psi}^{\pm\pm})$ . This multiplet has equal numbers of bosonic and fermionic excitations and hence gives a prefactor of zero for the Witten index.

The generators of the small  $\mathcal{N} = 4$  are four supercurrents  $\tilde{G}^{\pm\pm}$ , three bosonic currents  $J_R^i$  that generate a level  $k$   $SU(2)$  Kač-Moody algebra, and the usual Virasoro generators with central charge  $c = 6k$ . We have as usual two choices of boundary conditions for the fermionic operators, periodic (R) or antiperiodic (NS). The small  $\mathcal{N} = 4$  generators have certain commutation relations [52] with the fermionic operators  $\tilde{\psi}^{\pm\pm}$  and  $\tilde{J}^\varphi = \partial\varphi$ . We will only need the R-sector commutators

$$\{\tilde{G}_0^{\alpha a}, \tilde{\psi}_0^{\beta b}\} = \epsilon^{\alpha\beta}\epsilon^{ab}\tilde{J}_0^\varphi, \quad \{\tilde{G}_0^{\alpha a}, \tilde{J}_0^\varphi\} = \tilde{\psi}_0^{\alpha a}. \quad (59)$$

We wish to define an index which is nonvanishing for MSW. Consider the following table of the R sector ground states for the center of mass multiplet<sup>22</sup>

From this we see that a modified index that will not vanish due to the trivial contribution of the

<sup>22</sup>Here we have chosen the ground state to satisfy  $\tilde{\psi}_0^{-\pm}|0\rangle = 0$ .

state	$ 0\rangle$	$\tilde{\psi}_0^{\pm\pm} 0\rangle$	$\tilde{\psi}_0^{++}\tilde{\psi}_0^{--} 0\rangle$	total
$(-1)^F$	1	-1 ( $\times 2$ )	1	0
$\frac{1}{2}F^2(-1)^F$	0	-1	2	1

center of mass multiplet can be achieved by evaluating instead the trace of  $\tilde{F}^2(-1)^{\tilde{F}}$ , as shown in the third line of the table. It is an easy exercise to check that for states  $|s\rangle$  for which all  $\tilde{G}_0^{\alpha b}|s\rangle \neq 0$  the contributions from the various members of the supermultiplet cancel, so the modified trace defines a new index.

Interesting subtleties arise for the case in which only some of the supercharges annihilate the state. Consider for example a state that carries charges  $q_A \in \Gamma_M$

$$|q\rangle = e^{iD^{AB}q_A\phi_B}|0\rangle. \quad (60)$$

Acting with  $\tilde{G}_0^{\alpha a}$  on  $|q\rangle$  will produce states proportional to  $\tilde{\psi}_0^{\alpha a}|q\rangle$ , as can be checked using the commutation relations (59)

$$(\tilde{G}_0^{\pm\pm} - p^A q_A \tilde{\psi}_0^{\pm\pm})|q\rangle = 0. \quad (61)$$

This is simply the statement that  $|q\rangle$  preserves the supersymmetries nonlinearly, as we discussed before. As far as the index is concerned, the arithmetics is the same as for the Ramond ground states; in particular,  $\tilde{G}_0^{-a}|q\rangle = 0$  (in our random convention), and the contribution to the modified index is again 1.

Finally, we can refine our index by introducing potentials  $y^A$  for the  $h_{1,1}(M)$  conserved  $U(1)$  charges. The resulting index is called the *modified elliptic genus* of the MSW CFT

$$Z_{CFT}(\tau, \bar{\tau}, y^A) = \text{Tr}_R \frac{\tilde{F}^2}{2} (-1)^{\tilde{F}} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{2\pi i y^A q_A} \quad (62)$$

Here  $q = e^{2\pi i \tau}$  and the fermion number is

$$\tilde{F} = 2J_R^3 + p^A q_A. \quad (63)$$

If only RM ground states contributed to the index, then  $Z_{CFT}$  would be a holomorphic function of  $\tau$ , but due to the contribution of the states of

the form (60), some dependence on  $\bar{\tau}$  is also introduced, since

$$(\tilde{L}_0 - \frac{c_R}{24})|q\rangle = \frac{(p^A q_A)^2}{12D}|q\rangle. \quad (64)$$

Nevertheless, this dependence is entirely due to the RM boson in the center-of-mass multiplet, so it is under control. In fact, it can be shown [53] that the  $\bar{\tau}$  dependence of the elliptic genus is entirely of the form

$$Z_{CFT}(\tau, \bar{\tau}, y) = \sum_{\delta} Z_{\delta}(\tau) \Theta_{\delta}(\tau, \bar{\tau}, y) \quad (65)$$

where  $\Theta_{\delta}$  are lattice theta-functions and the sum has a finite number of terms.

To see how (65) comes about, recall that the  $h_{1,1}$ -dimensional lattice  $\Gamma_M$  is a sublattice of the larger  $6D$ -dimensional lattice  $\Gamma_P$  of  $h^{(3)}$  zero modes on the  $M5$ . Consider the unit cell  $\mathcal{U}$  of  $\Gamma_M$  in  $\Gamma_P$ , drawn in figure 6. Let  $\Gamma_{M+\delta}$  be a shift of the lattice  $\Gamma_M$  by a lattice vector  $\delta \in \mathcal{U}$ , and let  $\Gamma_{\perp M+\delta}$  be the lattice of vectors in  $\Gamma_P$  normal to vectors in  $\Gamma_{M+\delta}$  (this corresponds to directions in  $\Gamma_P$  perpendicular to the plane of the figure). It is clear that one can rewrite the partition sum in  $\Gamma_P$  as a sum over  $\Gamma_{M+\delta} \times \Gamma_{\perp M+\delta}$ , where  $\delta$  runs over the points in  $\mathcal{U}$ , a total of  $\det(6D_{AB})$  of them. The sum over  $\Gamma_{M+\delta}$  gives a  $\Theta$ -function that encodes the  $\bar{\tau}$  and  $y^A$  dependence, while the contribution from  $\Gamma_{\perp M+\delta}$  is clearly holomorphic. We note that (65) was used to derive an exact expression for the elliptic genus in some special cases [52].

At last, let us mention the existence of a spectral flow automorphism of the  $\mathcal{N} = 4$  algebra, which acts the following way on the bosonic generators:

$$\begin{aligned} \tilde{L}_0 &\rightarrow \tilde{L}_0 + \eta J_R^3 + \frac{c}{6} \eta^2 \\ J_R^3 &\rightarrow J_R^3 + \frac{c}{3} \eta. \end{aligned} \quad (66)$$

If the flow parameter  $\eta$  is integer the spectral flow acts on the NS and R sectors separately, while if  $\eta$  is half-integer, the NS and R sectors get interchanged. Hence the elliptic genus (62) can also be computed as a trace over the NS sector of the



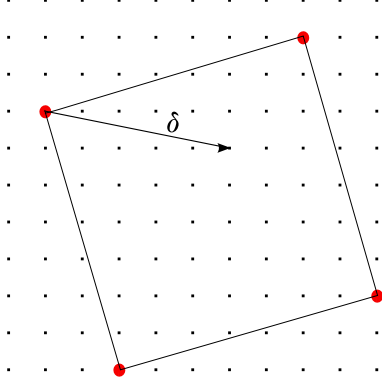


Figure 5. The unit cell of the lattice  $\Gamma_M$  embedded in  $\Gamma_P$ . The black dots are points in  $\Gamma_P$ , while the red dots are points in  $\Gamma_M$ . The sum over  $\delta$  runs over all points in the square, counting just one of the corners.

appropriately shifted variables

$$Z_{CFT}(\tau, \bar{\tau}, y^A) = \text{Tr}_{NS} \frac{\tilde{F}^2}{2} (-1)^{\tilde{F}} q^{L_0 - \frac{c}{24}} \times \bar{q}^{\tilde{L}_0 - \frac{1}{2} J_R^3} e^{2\pi i y^A q_A} \quad (67)$$

Now the contributions will come from the NS states that are related by spectral flow to the Ramond ground states. These are the *chiral primaries* and have  $\tilde{L}_0 = \frac{1}{2} J_R^3$ . Supergravity in  $AdS_3$  (or a quotient thereof) oftentimes produces NS boundary conditions for the fermions in the CFT living on the boundary torus. The trace over chiral primaries is then a more natural thing to compute from the supergravity perspective.

### 3. The topological string

In this lecture we briefly review what the A-model topological string is and computes, and describe the Gromov-Witten (GW) invariants. The many details omitted here can be found for example in [7, 54–58].

#### 3.1. Twisting the string

The topological string is obtained by twisting an  $\mathcal{N} = (2, 2)$  superconformal  $2d$  sigma-model

and coupling to  $2d$  gravity. We will start with just an untwisted  $(2, 2)$   $\sigma$ -model and build our way up to the A-model topological string. Due to the high amount of supersymmetry, the target space  $M$  of the  $\sigma$ -model is required to be Kähler. The action is then

$$S = 2t \int d^2 z (g_{i\bar{j}} \partial \phi^i \bar{\partial} \phi^{\bar{j}} + g_{i\bar{j}} \bar{\partial} \phi^i \partial \phi^{\bar{j}} + i g_{i\bar{j}} \psi_+^{\bar{j}} \nabla_{\bar{z}} \psi_+^i + i g_{i\bar{j}} \psi_-^{\bar{j}} \nabla_z \psi_-^i + R_{i\bar{j}k\bar{l}} \psi_+^{\bar{j}} \psi_+^{\bar{l}} \psi_-^k \psi_-^{\bar{l}})$$

where the coupling constant  $t$  sitting in front of the action can be thought of as  $\hbar^{-1}$ .

The  $\mathcal{N} = (2, 2)$  algebra has four worldsheet currents  $J, G^\pm, T$ , with spins  $1, \frac{3}{2}, 2$  (and their antiholomorphic counterparts, denoted by tilde).  $T$  is the energy-momentum tensor,  $J$  is the  $U(1)$   $R$ -symmetry current of the  $\mathcal{N} = 2$  algebra, and  $G^\pm$  are the conserved supercurrents for the two worldsheet supersymmetries; the  $\pm$  superscript denotes their  $R$ -charge. From the  $\mathcal{N} = 2$  algebra, the relations that are relevant to our discussion are the  $R$ -sector anticommutators

$$\begin{aligned} \{G_0^+, G_0^+\} &= \{G_0^-, G_0^-\} = 0 \\ \{G_0^+, G_0^-\} &= 2(L_0 - \frac{c}{24}) \end{aligned} \quad (68)$$

One can combine the two possible  $R$ -symmetries into a vector and an axial current  $J_V = J + \tilde{J}$ ,  $J_A = J - \tilde{J}$ . After quantising the theory, the quantum measure is invariant under the axial  $R$ -symmetry only if  $M$  is a Calabi-Yau manifold (of any dimension, so far), as it must satisfy

$$c_1(M) = 0. \quad (69)$$

Next, we would like to make our  $\sigma$ -model topological, which means that the observables in the theory should only depend on the topological data of the target space  $M$ . A way to accomplish this is by constructing a fermionic symmetry - generated by a (scalar<sup>23</sup>) BRST charge  $Q$ ,

<sup>23</sup>The reason that we need a scalar supercharge is that we would like our model to be defined for a worldsheet  $\Sigma$  of arbitrary curvature and genus. If  $Q$  is fermionic, then the supersymmetry parameter  $\epsilon$  has to be a Killing spinor on  $\Sigma$ , which in general does not exist. If  $Q$  is scalar though, then  $\epsilon$  is a Grassmann scalar, which exists for any worldsheet.

which is nilpotent - so that  $Q^2 = 0$ . The physical operators in the topological theory are then defined to be  $Q$  cohomology classes. Ultimately we will identify  $Q$  with a cohomology operator on  $M$ , such that the theory becomes topological. We further require the energy-momentum tensor to be  $Q$ -exact in order to ensure the independence of topological correlation functions of the  $2d$  worldsheet metric. All this is accomplished by twisting according to

$$T_{tws} = T - \frac{1}{2}\partial J, \quad (70)$$

and taking  $Q = \int G^+$ . This shifts the spin of the various operators by an amount proportional to their  $R$ -charge

$$s_{tws} = s - \frac{1}{2}q. \quad (71)$$

In particular, now  $J$  and  $G^+$  have spin 1, while  $T_{tws}$  and  $G^-$  have spin 2. Thus  $Q = \int G^+$  can act as our BRST charge, and  $T_{tws} \sim \{Q, G^-\}$ . Next we need to couple the  $\sigma$ -model to a worldsheet metric  $h_{\alpha\beta}$  and then perform the path integral over  $h_{\alpha\beta}$  too. In order to define string amplitudes, it is quite useful to note that the structure of the twisted  $\mathcal{N} = 2$  algebra is isomorphic to the structure we obtain by applying the BRST procedure to the usual bosonic string. The correspondence between the various operators is

$$(G^+, J, T, G^-) \leftrightarrow (Q, J_{ghost}, T, b). \quad (72)$$

Remember that if we want to compute genus  $g$  amplitudes in the bosonic string we need to integrate over the moduli space  $\mathcal{M}_g$  of genus  $g$  Riemann surfaces, with  $3g - 3$  insertions of the  $b$  ghost (for  $g > 1$ ), which provide the measure. In view of the correspondence between  $G^-$  and the  $b$  ghost, we define the genus  $g$  topological string amplitude as<sup>24</sup>

$$F_g = \int_{\mathcal{M}_g} \prod_{i=1}^{3g-3} \int_{\Sigma} d^2z G_{zz}^- \mu_i^z \bar{z} \int_{\Sigma} d^2z \tilde{G}_{\bar{z}\bar{z}}^- \mu_i^{\bar{z}} z$$

<sup>24</sup>For  $g = 0, 1$  see e.g. [55, 59, 7]. Note that the counterpart of the  $b$  ghost could be either  $\tilde{G}^+$  or  $\tilde{G}^-$ , depending on the twist we perform for the RM. Here we have made the choice corresponding to the A-model topological string.

$$(73)$$

where  $\mu_i$  are Beltrami differentials, parametris-ing complex structure deformations of the moduli space of genus  $g$  Riemann surfaces in  $M$ . It turns out that “ghost” charge conservation makes almost all  $F_g$  vanish, unless we take the complex dimension of  $M$  to be three. Therefore the case in which  $M$  is a Calabi-Yau three-fold provides the richest examples, which we will assume from now on.

The topological string “free energy” is defined as a perturbative expansion in the topological string coupling constant,  $g_{top}$

$$F_{top} = \sum_{g=0}^{\infty} g_{top}^{2g-2} F_g \quad (74)$$

The topological string partition function is

$$Z_{top} = e^{F_{top}} \quad (75)$$

Note that one could have equally considered a twist of the form  $T \rightarrow T + \frac{1}{2}\partial J$ . In that case  $G^-$  would become the BRST operator. Since we can do independent twists for the LM and for the RM, we end up with two inequivalent possible topological string theories, depending on which supercurrent becomes the BRST charge

$$\begin{aligned} (G^+, \tilde{G}^+) &\rightarrow \text{A model} \\ (G^+, \tilde{G}^-) &\rightarrow \text{B model} \end{aligned}$$

As advertised, in these notes we will only concentrate on the A-model topological string.

### 3.2. The A-model topological string

The A-model twist is

$$L_0 \rightarrow L_0 - \frac{1}{2}J_0, \quad \tilde{L}_0 \rightarrow \tilde{L}_0 + \frac{1}{2}\tilde{J}_0 \quad (76)$$

which shifts the spins as

$$S \rightarrow S - \frac{1}{2}(J_0 + \tilde{J}_0). \quad (77)$$

The local observables in the theory are in one-to-one correspondence with the de Rham cohomology classes on  $M$ . Correlation functions only receive contributions from configurations that satisfy the classical equations of motion, which in the

A-model are holomorphic maps from the string worldsheet to the target space. The action can be written as a  $Q$ -exact term (which does not contribute) plus

$$\Delta S = -it \int_{\Sigma} \phi^*(J) = -it \int_{\phi(\Sigma)} J = q_A t^A \quad (78)$$

where  $q_A$  are the wrapping numbers of the image of the string worldsheet  $\phi(\Sigma)$  around the various 2-cycles dual to  $\omega_A$  and  $t^A$  are the complexified Kähler moduli of the Calabi-Yau. We also set  $t = (\alpha')^{-1}$ . To compute the topological string partition function, we need to integrate over the moduli space  $\mathcal{M}_{g,\mathbf{q}}$  of maps from a genus  $g$  Riemann surface to the Calabi-Yau  $M$  - where the image belongs to the homology class  $\mathbf{q}$  of  $M$ , weigh it by the exponential of minus the euclidean action (78), and then sum over genera. From (78) it is clear that the A-model topological string only depends on half the information of the Calabi-Yau - that is - only on the Kähler structure.

The *Gromov-Witten* invariants  $d_{g,q_A}$  are defined as the expansion coefficients in

$$F_{GW}(t^A) = \sum_{g=0}^{\infty} g_{top}^{2g-2} \sum_{q_A} d_{g,q_A} e^{-q_A t^A} \quad (79)$$

which encode the contributions which are nonperturbative in  $\alpha'$ .  $d_{g,q_A}$  is roughly the euler character of  $\mathcal{M}_{g,q_A}$ .

There are also perturbative contributions to  $F_{top}$ , that are not encoded in  $F_{GW}$ . These contributions can be computed in just low-energy field theory, and are only present at genus 0 and 1

$$F_{pert}(t^A) = -i \frac{(2\pi)^3}{g_{top}^2} D_{ABC} t^A t^B t^C - \frac{i\pi}{12} c_{2A} t^A. \quad (80)$$

#### •What the topological string computes

Recall from our discussion in section (2.7) that higher curvature corrections to the  $\mathcal{N} = 2$  supergravity action in 4d are encoded in a holomorphic function of the vector moduli and the Weyl mul-

tiplet

$$\begin{aligned} S_{corr} &\sim \int d^4x d^4\theta \mathcal{F}_g(X^\Lambda) W^{2g} + c.c. \\ &\sim \int d^4x [F_g(t) R_-^2 T_-^{2g-2} + F_g(\bar{t}) R_+^2 T_+^{2g-2}] \end{aligned} \quad (81)$$

where  $T_{\pm}$  and  $R_{\pm}$  are the self-dual and anti-self-dual parts of the graviphoton and Weyl tensor respectively. In usual string theory, such a contribution to the effective action can be computed by evaluating the correlation function of two graviton and  $2g-2$  graviphoton vertex operators on a genus  $g$  string worldsheet. This computation has been performed in [60] and it was found that the amplitude for this scattering process is precisely given by the genus  $g$  topological string amplitude  $F_g$ .

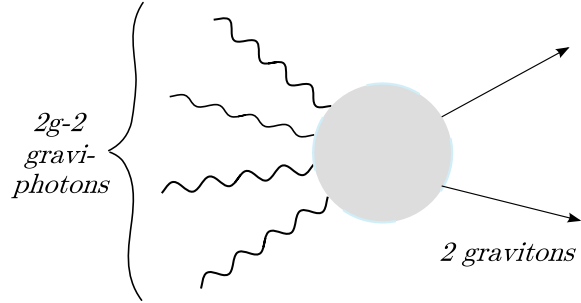


Figure 6. Scattering of two gravitons and of  $2g-2$  graviphotons gives  $F_g$ .

#### 4. The OSV conjecture

In lecture 2 we discussed black hole attractors in the context of type IIA string theory compactified on a Calabi-Yau  $M$ . We found that the entropy of these black holes only depended on the Kähler moduli of the compactification, which in turn are fixed to their attractor values at the horizon of the black hole. Also, remember that hypermultiplet scalars - which in type IIA correspond to the complex structure deformations of

$M$  - completely dropped out of the story. Interestingly, in the A-model topological string half of the information on the Calabi-Yau data also drops out due to the twist. This is a first clue towards a connection between attractor black holes and topological strings: they are both functions on half of the Calabi-Yau moduli space. It would be strange if in string theory we had two natural functions of the same variables that were not related in some simple way. Indeed we shall see that, when the comparison between the two quantities is properly formulated, the relation between the indexed black hole entropy and topological string partition functions appears to take the simplest imaginable form.

#### 4.1. Mixed partition functions

If we want to make an exact or all orders comparison between the partition functions of two systems, we must first be precise about exactly which ensemble we are using. A statistical system or ensemble can be characterized either by conserved quantities such as the energy  $E$  or by conjugate potentials such as the inverse temperature  $\beta$ . There are associated microcanonical and canonical partition functions  $\Omega$  and  $Z$ , which for the case of energy and temperature are related by

$$Z(\beta) = \sum_E e^{-\beta E} \Omega(E). \quad (82)$$

More generally, if there are many conserved charges/potentials, partition functions can be defined in which some variables are treated canonically and others microcanonically. We refer to these as *mixed* partition functions associated to *mixed* ensembles.

For every type of (mixed) ensemble there is an associated definition of entropy. For example, in the microcanonical ensemble the entropy is defined as

$$S_E(E) = \ln \Omega(E), \quad (83)$$

while in the canonical ensemble it is defined by

$$S_\beta(\beta) = \ln Z - \beta \partial_\beta \ln Z. \quad (84)$$

To leading order in the saddle point approximation one has

$$S_\beta(\beta(E)) \sim S_E(E) \quad (85)$$

with  $\beta(E) \equiv \partial_E S_E$ . However in general, for a finite system and beyond leading order,  $S_\beta$  and  $S_E$  are not related as simply as in (85) - by just a change of variables. Rather, an integral transform is needed. Hence if we want to discuss subleading corrections to the entropy we must specify exactly which ensemble we are using to define the entropy. It is meaningless to give an all-orders formula for the entropy without this specification.

So the question arises: exactly which entropy does Wald's formula, discussed in section 2.7.2, compute? Wald's derivation is ultimately based on a quantum field theory analysis. The answer then follows from the fact that in quantum field theory the boundary conditions amount to an implicit choice of ensemble. For example, if we sum over geometries with asymptotic periodicity  $\beta$ , we are working in the canonical fixed-temperature ensemble. There is no field theory path integral formulation for the microcanonical ensemble. The best we can do is a Laplace transform

$$\Omega(E) = \int d\beta e^{\beta E} Z(\beta), \quad (86)$$

which requires knowledge of  $Z(\beta)$  for all  $\beta$ . So for a finite-temperature black hole Wald's formula computes  $S_\beta$  rather than  $S_E$ .

What about charges? For electromagnetic charges, instead of fixing the radius of the circle at infinity, in the field theory path integral one fixes the boundary value of the field  $A_0 = \phi$  (the electric Wilson line) at infinity, as well as the topological class of the gauge field. This corresponds to fixing the magnetic charge  $q_m$  while summing over all electric charges with weights  $e^{-\phi q_e}$ . Hence field theory gives a mixed ensemble which treats electric charges canonically and magnetic charges microcanonically. This then is the ensemble in which the entropy computed by Wald's formula is defined<sup>25</sup>. We will see this is crucial for comparison of Wald's formula for the entropy to the topological string partition function.

We note that an independent but similar issue is whether or not the ensemble is weighted

<sup>25</sup>See also [61] for a discussion of thermodynamic ensembles.

with minus signs for fermions. This is again related to path integral boundary conditions and is discussed below.

#### 4.2. Brute force derivation of OSV

Now we would like to get to work and compute the corrections to the entropy, using Wald's formula. The first question is, which terms from the corrected  $\mathcal{N} = 2$  action do actually contribute. The working assumption has been [44]<sup>26</sup> that if we compute the indexed entropy - defined in the usual way as a Legendre transform of the supersymmetry-protected indexed partition function (more precisely, the modified elliptic genus (62)) that there is a (perturbative) nonrenormalization theorem implying that it receives contributions only from supersymmetry-protected terms in the effective action. These are the terms which involve integrals over only a chiral half of superspace and so cannot involve hypermultiplets. Hence we expect that only terms in the action that are built exclusively out of vector multiplets can correct the entropy. These are given of course by (82), and their contribution to the entropy has been quoted in section 2.7

$$S_{BH} = \frac{\pi i}{2}(q_\Lambda \bar{C} \bar{X}^\Lambda - p^\Lambda \bar{C} \bar{F}_\Lambda) - \frac{\pi}{2} \text{Im}(C^2 W \partial_W F). \quad (87)$$

Naively, one might evaluate the  $X^\Lambda, F_\Lambda$  at their attractor values (51) and (52) and view this as an expression for the entropy as a function of the magnetic and electric charges  $p^\Lambda$  and  $q_\Lambda$ . However, according to the discussion of the previous section, this is not correct: (87) comes from a mixed ensemble and should be viewed as a function of the magnetic charges  $p^\Lambda = \text{Re} C X^\Lambda$  and the electric potentials, which shall be identified shortly as  $\phi^\Lambda = \pi \text{Im} C X^\Lambda$ .

The above expression for the entropy can be rewritten in a nicer form by using the homogeneity property of the prepotential<sup>27</sup> and the attrac-

tor equations

$$S_{BH}(p, \phi) = -\pi \text{Im} F(C X^\Lambda, 256) + \pi \text{Im}(C X^\Lambda) q_\Lambda, \quad (88)$$

which let us eliminate  $F$  and  $F_\Lambda$  in favor of  $p^\Lambda$  and  $\phi^\Lambda$ . In terms of  $\phi^\Lambda$  and the imaginary part of  $F$  this is

$$S_{BH} = \mathcal{F}(p^\Lambda, \phi^\Lambda) + \phi^\Lambda q_\Lambda, \quad (89)$$

where

$$\mathcal{F}(p, \phi) = -\pi \text{Im} F(p^\Lambda + \frac{i}{\pi} \phi^\Lambda, 256). \quad (90)$$

The second half of the attractor equations then reads

$$q_\Lambda = -\frac{\partial}{\partial \phi^\Lambda} \mathcal{F}(p, \phi) \quad (91)$$

which implies that  $S_{BH}$  is obtained from  $\mathcal{F}(p, \phi)$  in exactly the same way that the entropy is obtained from the logarithm of the mixed partition function. Namely,  $S_{BH}$  can be thought of as the Legendre transform with respect to the canonical variables only

$$S_{BH}(q, p) = \mathcal{F}(\phi, p) - \phi^\Lambda \frac{\partial}{\partial \phi^\Lambda} \mathcal{F}(\phi, p) \quad (92)$$

provided that the  $\phi^\Lambda$  are indeed identified as the conjugate potentials to the electric charges  $q_\Lambda$ .

In conclusion,  $\mathcal{F}(p, \phi)$  is the logarithm of the partition function computed in a mixed ensemble, in which one fixes the magnetic charges and electric potentials at infinity. We will loosely refer to  $\mathcal{F}$  as the free energy though strictly speaking it differs by a factor of  $-\beta$  from the usual definition.

One relatively illuminating way to write the mixed free energy is by picking a gauge in which  $C = 2Q$ , where  $Q$  is the graviphoton charge. Then, using the homogeneity properties of  $F$ , we have

$$\mathcal{F} = -4\pi Q^2 \text{Im} \left[ \sum_g F_g \left( \frac{p^\Lambda + i\phi^\Lambda/\pi}{2Q} \right) \left( \frac{8}{Q} \right)^{2g} \right] \quad (93)$$

<sup>26</sup>Reference [44] does not address all the issues involved and this point remains in need of further clarification. Some relevant, but still incomplete, observations are made section 4.3.1 below.

<sup>27</sup> $2F = X^\Lambda F_\Lambda + W \partial_W F$ .

which shows very clearly the perturbative nature of  $Z_{BH}$  as an expansion around large graviphoton charge. Now, the topological string partition function is also defined by a perturbative expansion in  $g_{top}$  (74) and, as mentioned in section 2.7, it is proportional to the supergravity prepotential  $F(CX^\Lambda, 256)$ . By comparing the first two terms (80) in the expansions of the two prepotentials, one can fix all normalizations and finds

$$F(CX^\Lambda) = -\frac{2i}{\pi} F_{top}(t^A, g_{top}) \quad (94)$$

with the following correspondence between the arguments on the two sides of the equation<sup>28</sup>

$$CX^\Lambda = p^\Lambda + i\phi^\Lambda/\pi, \quad t^A = \frac{CX^A}{CX^0}, \quad g_{top} = \pm \frac{4\pi i}{CX^0} \quad (95)$$

From (94) one finds

$$\ln Z_{BH} = -\pi \text{Im} F = 2 \text{Re} F_{top} \quad (96)$$

which can be written in the more expressive form

$$Z_{BH}(\phi^\Lambda, p^\Lambda) = |Z_{top}(t^A, g_{top})|^2 \quad (97)$$

with the variables identified as in (95). As there are several assumptions that went in to the derivation of (97) it is known as the OSV conjecture. It is important to note that the conjecture is a statement about the equality of two perturbation expansions. Indeed at the nonperturbative level it is not clear how either side of (97) is even defined.

Note that the factor of ‘2’ in the exponent of (97) was obtained by the brute force method – just carefully keeping track of all normalizations. So far there is no hint as to why it should not equal 1 or 17 or any other real number. Such a simple relation demands a simple physical explanation, which we will provide in the next section.

<sup>28</sup>For a D4-D2-D0 black hole, in the large-charge, large Calabi-Yau volume approximation,  $g_{top} = \frac{4\pi^2}{\phi^0} \sim \sqrt{\frac{q_0}{D}} \ll 1$  if we uniformly scale up the charges of the black hole. This is consistent with the fact that we are performing a perturbative expansion in the topological string coupling constant.

### 4.3. Why $Z_{BH} = |Z_{top}|^2$

In this section we will give a heuristic explanation of why the two complex conjugate factors  $Z_{top}$  and  $\bar{Z}_{top}$  appear in the black hole partition function. The goal here is not to give a perturbative proof of the OSV conjecture – many points would have to be filled in and clarified – but rather to provide a compelling physical picture<sup>29</sup>.

The basic idea is simply to evaluate the black hole partition function in a string perturbation expansion around a euclidean saddle point, which is a euclideanized attractor geometry (and then compare it to the perturbation expansion for the topological string). It is then argued that at  $g$  loops, string perturbation theory is saturated by genus  $g$  worldsheet (anti)instantons which wrap (anti)holomorphic cycles in the Calabi-Yau and localize to the (south) north pole of the horizon  $S^2$ . This is possible due to the peculiarities of the  $AdS_2 \times S^2$  supersymmetries, which are broken in exactly the same way for an instanton at the north pole and an anti-instanton at the south pole. The two factors in  $Z_{BH}$  then come from the instanton sum at the north pole and the anti-instanton sum at the south pole.

#### 4.3.1. M-theory lift

Our starting point will be the IIA string partition function – denoted  $Z_{IIA}$  – on the euclidean attractor geometry. As will be seen explicitly below, the sum over worldsheets gives a finite non-zero answer. Furthermore, there must be many supersymmetric cancellations, as in the limit in which the  $AdS_2 \times S^2$  radius goes to infinity the term which scales as volume is the cosmological constant and must vanish.

So the question arises – what is  $Z_{IIA}$  computing? Since the computation involves a choice of attractor geometry, it should be related to the associated black hole. Since it is supersymmetry protected, it should be some supersymmetry-protected invariant associated with the black hole. We know of only one such object – the mod-

<sup>29</sup>We follow here [62], which in turn built on earlier discussions in [63–65, 22].

ified elliptic genus. So the obvious guess is

$$Z_{IIA} = Z_{CFT} + \mathcal{O}(e^{-\frac{1}{g}}). \quad (98)$$

Note that since the left hand side is defined only in string perturbation theory, this is at best a perturbative relation (at large charges). If one further assumes  $Z_{BH} = Z_{CFT} + \mathcal{O}(e^{-\frac{1}{g}})$  one has  $Z_{IIA} = Z_{BH}$ .

There are a number of ways one might go about demonstrating (98). The most straightforward approach would be to refine and adapt the methods of [66,67]. The subtleties would be to carefully understand the euclideanization, especially of the RR fields which become complex, and the fermion boundary conditions. Instead of this more direct method, [62] adopted a shortcut involving a lift to M-theory. This had the disadvantage of using a nonperturbative relation to demonstrate a perturbative one, but on the other hand the construction, which we now review, is illuminating in its own right.

To keep the equations uncluttered, we just consider the euclideanised near-horizon geometry of a D4-D0 black hole. The metric on the  $AdS_2 \times S^2$  part is

$$ds_4^2 = 4l^2 \left( \frac{dr^2 + r^2 d\theta^2}{(1-r^2)^2} + \frac{1}{4} d\Omega_2^2 \right), \quad (99)$$

the radii of the  $AdS_2$  and  $S^2$  are equal to

$$l = \frac{g_s \phi^0}{\pi} \sqrt{\alpha'} = g_s \sqrt{\frac{D\alpha'}{q_0}} + \text{corrections} \quad (100)$$

There is also a graviphoton field

$$A^{(1)} = -\frac{2i\phi^0}{\pi} \frac{d\theta}{1-r^2}. \quad (101)$$

Note that this potential is purely imaginary - as expected when we continue electric fields to euclidean space. Noting that the graviphoton is just the connection on the M-theory circle (parametrized here by  $x^{11}$ ), we simply end up with the M-theory metric

$$ds_{11}^2 = g_s^2 \alpha' \left( dx^{11} - \frac{2i\phi^0}{\pi} \frac{d\theta}{1-r^2} \right)^2 + 4l^2 \frac{dr^2 + r^2 d\theta^2}{(1-r^2)^2} + l^2 d\Omega_2^2 + ds_{CY}^2. \quad (102)$$

We immediately recognise an  $AdS_3$  factor in the first line of the above equation. Note that some of the metric components are imaginary, so this is a *complexified*  $AdS_3$  quotient. The interpretation of this complexified geometry can be gleaned from looking at the torus that is located at the boundary  $r = 1$ . Keeping the leading terms in the metric as we take  $\epsilon = 1 - r \rightarrow 0$ , we find the conformal metric on the boundary

$$ds_{bnd}^2 = d\theta^2 + \frac{2\pi i}{\phi^0} d\theta dx^{11} + \mathcal{O}(\epsilon). \quad (103)$$

This can be put in the standard form

$$ds_\tau^2 = |d\theta + \tau dx^{11}|^2 \quad (104)$$

if we take the modulus of the torus parametrised by  $(\theta, x^{11})$  to satisfy

$$|\tau|^2 = (Re\tau)^2 + (Im\tau)^2 = 0 \text{ and } Re\tau = i \frac{\pi}{\phi^0}. \quad (105)$$

Now, according to  $AdS_3/CFT_2$ , the partition function on this euclidean M-geometry is a partition function of the MSW CFT on the boundary torus (103)

$$\begin{aligned} Z_M(\tau) &\sim Tr_{MSW} e^{2\pi i Re\tau(L_0 - \bar{L}_0)} e^{-2\pi Im\tau(L_0 + \bar{L}_0)} \\ &\sim Tr_{MSW} e^{-\frac{4\pi^2}{\phi^0} L_0}. \end{aligned} \quad (106)$$

The main point we wish to stress here is that  $\tau$  appears as a weighting factor for the left movers only. This result might have been anticipated from the fact that the lorentzian D4-D0 black hole is dual to a state in the MSW CFT in which the left movers are thermally excited but the right movers are in their ground state.

Consideration of the fermion boundary conditions [62] indicates that, after a spectral flow one obtains (106) with the trace in the R sector and an insertion of  $(-)^F$ . An  $F^2$  insertion is needed when the degrees of freedom corresponding to the center-of-mass multiplet (ignored so far) are included: without the insertion this multiplet gives a zero prefactor. In the bulk  $AdS$  description these appear as boundary “singleton” modes. Exactly how this all works out has not been carefully

analyzed and the  $F^2$  insertion is accordingly suppressed in the following.

Of course, the index is defined for fixed electric potentials on the boundaries, so we need to add D2-brane charges, and then sum over them. Upon reinstating the D2 potentials  $\phi^A$ , the partition function we are computing becomes

$$Z_{IIA}(\phi^0, \phi^A, p^A) \sim \text{Tr}(-1)^F e^{-\frac{4\pi^2}{\phi^0} L_0 - q_A \frac{\phi^A}{\phi^0}}, \quad (107)$$

where here and elsewhere  $\mathcal{O}(e^{-1/g})$  corrections are implicit.

In order to make the connection with the topological string, it is useful to rewrite (107) in terms of topological string variables. Using (95) for the D4-D2-D0 black hole, we can rewrite (107) as

$$\begin{aligned} Z_{IIA} &= \text{Tr}(-1)^F e^{-g_{\text{top}} L_0 - q_A (t^A + i\pi p^A)} \\ &= \text{Tr}(-1)^{\tilde{F}} e^{-g_{\text{top}} L_0 - q_A t^A} \end{aligned} \quad (108)$$

where  $\tilde{F}$  is just the modified RM fermion number (63) needed for modular invariance and  $q_A t^A$  is the correct string instanton action.

In conclusion, string theory on the euclidean attractor geometry is expected to give a perturbative expansion of the supersymmetric partition function of the associated black hole.

#### 4.3.2. Computing $Z_{IIA}$

Now we must evaluate the perturbative string loop expansion of the type IIA partition function on the attractor geometry. This computation could be set up in the NS-R, Green-Schwarz or hybrid formalism, each of which has its own complementary set of advantages and disadvantages. In the NS-R formalism the connection to the topological string is clearest, but RR fluxes are hard to deal with. In the Green-Schwarz formalism the action in RR backgrounds is known, but the reduction to the topological string is only partially worked out [68]. The hybrid formalism, as it more or less treats the  $4d$  spacetime part in the Green-Schwarz language and the Calabi-Yau part in the NS-R language is perhaps ultimately the most suitable, but unfortunately it is less developed at present.

In [62] we have used the Green-Schwarz formalism. The worldsheet action effectively splits into an internal  $CY_3$  term and an external  $AdS_2 \times S^2$  piece, and the computation factorizes. The details of exactly how the internal piece reproduces the topological string have not all been completed, but the presumed equivalence of the Green-Schwarz and NS-R strings imply they must work out, as shall be assumed herein. Although of interest in their own right these details are essentially the same in  $CY_3 \times \mathbb{R}^4$  and in an attractor geometry and are not the focus of our current investigation. Here we focus on the  $AdS_2 \times S^2$  factor where interesting new features arise.

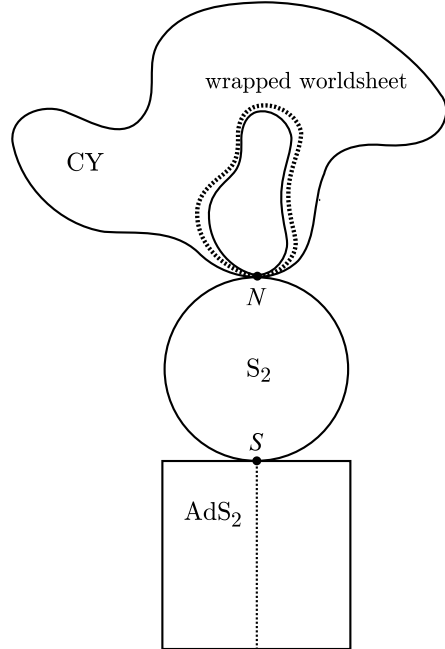


Figure 7. Illustration of a contributing worldsheet instanton, which wraps a curve in the Calabi-Yau, sits at the north pole of the  $S^2$  and the center of  $AdS_2$ . Anti-instantons wrap the same curve but with the opposite orientation, sit at the south pole and the center of  $AdS_2$ .

Let us now summarize the key steps in the



somewhat technical computation [62]. The classically supersymmetric genus  $g$  worldsheets  $\Sigma_g$  wrap holomorphic cycles (instantons) or antiholomorphic cycles (anti-instantons) in  $M$  and sit at any point in  $AdS_2 \times S^2$ . However, as will be important momentarily, which supersymmetry is preserved depends on the point chosen in  $AdS_2 \times S^2$ . The worldsheet sum is organized as an expansion about these configurations. The internal part (presumably) gives a factor of  $F_g(CX^A)$ , as we will try to sketch below. Afterwards, we are just left with the ordinary integral of a (-1)-brane zero-mode action in  $AdS_2 \times S^2$ .

The “fields” consist of the four spacetime coordinates  $X^\mu$ , their four chiral goldstino superpartners  $\theta_{1,2}^\alpha$ ,  $4g$  fermionic fields  $\rho_i^\alpha$  coming from the  $4g$  zero modes of the canonically conjugate momenta to the  $\theta_{1,2}^\alpha$  on  $\Sigma_g$ , and possibly some antichiral fields  $\chi^{m\dot{\alpha}}$ ,  $\chi^{\bar{m}\dot{\alpha}}$ . The latter correspond to zero modes of the fermionic superpartners of the normal fluctuations of the worldsheet  $\Sigma_g$  inside the Calabi-Yau, which occur whenever  $\Sigma_g$  is not an isolated curve in  $M$ .

There is no action for  $X^\mu$  or  $\theta_{1,2}$ . The action for  $\rho^\alpha$  is simply [68]

$$S_{int} = \int W_{\alpha\beta} \rho^\alpha \wedge \rho^\beta, \quad (109)$$

$W_{\alpha\beta}$  is the anti-self-dual part of the graviphoton field strength, with attractor value  $g_{top}$ . Thus the  $\rho$  contribution to the path integral is

$$\int d^{4g} \rho e^{-\int W \rho^2} = \int d^{4g} \rho (W \rho^2)^{2g} = g_{top}^{2g}. \quad (110)$$

The fermionic zero-modes  $\chi^{m,\bar{m}}$  and their bosonic superpartners are described by supersymmetric quantum mechanics on the moduli space  $\mathcal{M}_{g,qA}$  of holomorphic deformations of the curve inside  $M$ . The supersymmetric index we are computing is independent of the details of this quantum mechanics and is known to equal the euler character<sup>30</sup> of  $\mathcal{M}_{g,qA}$ , which is roughly the

<sup>30</sup>A typical four-supercharge action contains a coupling  $\int R_{m\bar{n}p\bar{q}} \chi^{m\dot{\alpha}} \chi^{\bar{n}\dot{\beta}} \chi^p_{\dot{\alpha}} \chi^{\bar{q}}_{\dot{\beta}}$ , and one can bring down powers of this interaction to make the path integral over the  $\chi^m$  zero modes not vanish. One can easily check that this procedure yields the euler character of moduli space.

Gromov-Witten invariant  $d_{g,qA}$ . Combining it with the factor that comes from the instanton action,  $e^{-qA t^A}$ , we finally obtain  $F_{g,qA}(t^A)$ .

It remains to consider the integral over the position zero-mode  $X^\mu$  and the goldstinos  $\theta_{1,2}$ . The former is proportional to the volume of  $AdS_2 \times S^2$  while the latter vanishes because there is no action. Hence

$$F_4 = \int d^4 x d^2 \theta_1 d^2 \theta_2 e^{-0} = \infty \times 0 = ?! \quad (111)$$

This integral can be defined by use of localization. One adds an exact term  $\Delta S = \delta K$  to the action with an arbitrary real coefficient  $t$

$$F_4 = \int d^4 x d^2 \theta_1 d^2 \theta_2 e^{-t \Delta S}. \quad (112)$$

$\delta$  here is a nilpotent combination of the kappa-symmetries and supersymmetries and  $K$  is a judiciously chosen operator. Exactness and nilpotency imply that the integral is independent of  $t$ . It is most easily evaluated at  $t \rightarrow \infty$ . The contributions will then be localized to  $\delta$ -invariant configurations, which are instantons and anti-instantons sitting at the center of  $AdS_2$  and the north pole and respectively south pole of the sphere. The answer we get is some nonzero constant  $C$ , whose value is most easily determined by comparison to supergravity.

Now let us put everything together. Concentrating for now just on the contribution of the instanton at the north pole, we have the following

- a genus-independent constant factor  $C$  coming from the integral over the bosonic and goldstino zero-modes
- a factor of  $F_g(t^A)$  coming from the internal worldsheet partition function.
- a factor of  $g_{top}^{2g}$  from the zero-mode integrals for  $\rho^\alpha$ .

Summing these over genera we obtain

$$\sum_g C F_g g_{top}^{2g} = g_{top}^2 C F_{top}. \quad (113)$$

The anti-instanton sitting at the south pole of the sphere similarly gives a factor of  $g_{top}^2 \bar{C} \bar{F}_{top}$ .

Exponentiating the sum of the two contributions we almost have the OSV relation, except we still need to find the value of  $C$ . This can be fixed by comparing any term in the expansion of  $Z_{BH}$  with the corresponding term in the topological string expansion, as was done in section 3.6, and is  $C = g_{top}^{-2}$ . In conclusion this indicates that in perturbation theory

$$Z_{BH} = Z_{IIA} = |Z_{top}|^2$$

which is the relation we set out to explain.

## 5. $Z_{BH}$ , $Z_{CFT}$ , $Z_{IIA}$ , $Z_{top}$ and all that

In these lectures several closely related partition functions, all denoted with the letter “ $Z$ ”, have appeared. We will close by summarizing what we mean by these objects, how they are computed (or not) and how they are related.

### 5.1. Definitions

• **$Z_{BH}$**  This partition function is deduced from a perturbative spacetime analysis of the thermodynamic properties of black holes, which at leading order are governed by the famous Bekenstein-Hawking area-entropy law. Subleading corrections are computed using Wald’s formula. We are not using this symbol herein to denote an object defined nonperturbatively by counting BPS states.

In our case, we are interested in extremal black holes, and the corrections to the area law are computed in a power series in the inverse graviphoton charge  $Q$ , which governs the size of the black hole. Moreover we are interested in a supersymmetric partition function with (among other things) a  $(-)^F$  insertion for fermions. In principle the effect of this on the spacetime analysis and Wald’s formula should be derivable from first principles, but in practice no one has done so. The arguably well-motivated assumption has been that the insertions are accounted for by including only  $F$  term corrections to Wald’s formula.

Any definition of  $Z_{BH}$  along these lines will necessarily entail some dependence on the moduli of the Calabi-Yau compactification in which the black hole sits. This dependence has a beau-

tiful and intricate mathematical structure following from the “split attractor” phenomena, the unraveling of which has been the subject of recent progress [22].

This definition of  $Z_{BH}$  is essentially perturbative in  $\frac{1}{Q}$ , and does not have any obvious non-perturbative completion.

• **$Z_{CFT}$**  This is defined as the modified elliptic genus, with appropriate potentials, of the MSW CFT. In principle this has a chance of being defined nonperturbatively. However there are a number of complicating issues which have not been understood (see e.g. [53]). An important one is that one must regulate IR divergences associated with noncompact Coulomb branches - where the M5-branes can separate. Related issues are the holomorphic anomaly, background dependence and singularities in the moduli space of divisors. This is the only potentially nonperturbatively defined object on our list.

• **$Z_{IIA}$**  This is the IIA partition function computed in the string genus expansion around a euclidean attractor geometry. This expansion is formally equivalent to an expansion in the RR graviphoton field, so the problems of dealing with background RR fields are not so severe here. There are the usual issues associated with euclidean quantum gravity and complex saddle points that must be dealt with.

• **$Z_{top}$**  This is the topological string partition function, and is defined in the worldsheet genus expansion. The expansion coefficients have a precise mathematical definition in terms of the Gromov-Witten invariants associated to maps of a genus  $g$  Riemann surface into a Calabi-Yau space. So  $Z_{top}$ , unlike  $Z_{IIA}$  and  $Z_{BH}$ , is fully defined in perturbation theory. No nonperturbative definition of the full function  $Z_{top}$  is known.

• **$\Omega(p, q; \tau_\infty)$**  Another interesting object, not directly discussed in these lectures but ultimately very relevant is  $\Omega(p, q; \tau_\infty)$ . This is the (weighted) number of BPS states with charges  $(p, q)$  which depends (due to jumping phenomena) on the asymptotic value  $\tau_\infty$  of the Calabi-Yau moduli, and should be nonperturbatively well defined. A recent and mathematically precise definition of  $\Omega$ ,

a derivation of some of its properties and a corresponding precise version of the strong *OSV* conjecture can be found in [22]. In our list of objects above, only  $Z_{CFT}$  is non-perturbatively defined and so potentially related to  $\Omega$ .

## 5.2. Relations

At a very formal level,  $Z_{BH} = Z_{IIA} = Z_{CFT} = |Z_{top}|^2$ , in the appropriate perturbation expansions and with the appropriate identifications of parameters. Beyond perturbation theory, these relations are not even formally true. In practice, none of these equalities have been proven or even stated herein with total precision. Let us now comment on the various equalities.

### • $Z_{BH} = Z_{CFT}$

This is a deep statement about the holographic nature of string theory, and was the early form of *AdS/CFT* duality [69]. At leading order it has been tested in many examples. Some tests of subleading terms are discussed in [46,48]. Beyond subleading order a better understanding of the definition of the left hand side is needed.

### • $Z_{BH} = |Z_{top}|^2$

This is the weak form of the *OSV* conjecture. It equates terms in two differently defined perturbation expansions [52,65,70–76]. One of the main issues here is understanding how the background dependence works out. In essence however it is a statement about string perturbation theory and does not have the dynamical depth of the preceding equality. It could in principle be precisely stated and proved without any understanding of non-perturbative string theory.

### • $Z_{BH} = Z_{IIA}$

This asserts that the black hole partition function can be perturbatively repackaged as a euclidean calculation on the attractor geometry. It is a stringy version of the black hole methods of semiclassical euclidean quantum gravity, adapted to deal with an index. Arguments for this equality were given in [62]. Assuming that this and the preceding equality is valid for the same definition of  $Z_{BH}$  leads to  $Z_{IIA} = |Z_{top}|^2$ , along with a physical explanation of the two complex conjugate factors on the RHS of the *OSV* relation.

### • $Z_{CFT} = |Z_{top}|^2$

This is the strong form of the *OSV* conjecture, and comes from combining holographic duality with the perturbative observations of [3]. Since the right hand side is defined only perturbatively, it is at most a perturbative statement. Still, it is extremely interesting in part because both sides are potentially rigorously defined mathematically. From the mathematical point of view, it is a totally unexpected relationship between moduli spaces of divisors and maps of curves into a Calabi-Yau space. In this paper we have argued from the stringy perspective that a relation of this form is to be expected. It would be a daunting task to turn these arguments in to a rigorous mathematical derivation. While these arguments are informative, in the end this relation should be regarded as a conjecture to be tested by direct computation.

In conclusion, the conjectured *OSV* relationship between the all-orders expression for the entropy of large BPS black holes and the all-orders expression for the topological string partition function potentially allows for precision tests of non-perturbative string theory. We hope to have provided the reader with a flavor of this exciting subject that melts together black holes, attractors, and topological strings, and at the same time raises interesting new challenges and puzzles.

## • Acknowledgements

We are grateful to F. Denef, D. Gaiotto, G. Moore and Xi Yin for very helpful conversations. Special thanks to C. García for help with the figures. Finally we wish to thank Laurent Baulieu, Pierre Vanhove, Paul Windey, Mike Douglas, Jan de Boer and Eliezer Rabinovici for a really stimulating and fun school. This work has been partially supported by DOE grant DE-FG02-91ER40654.

## REFERENCES

1. B. Bates and F. Denef, “Exact solutions for supersymmetric stationary black hole composites,” [hep-th/0304094](#).

2. H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, “A supersymmetric black ring,” *Phys. Rev. Lett.* **93** (2004) 211302, [hep-th/0407065](#).
3. H. Ooguri, A. Strominger, and C. Vafa, “Black hole attractors and the topological string,” *Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146](#).
4. D. Gaiotto, A. Strominger, and X. Yin, “New connections between 4D and 5D black holes,” *JHEP* **02** (2006) 024, [hep-th/0503217](#).
5. G. W. Moore, “Arithmetic and attractors,” [hep-th/9807087](#).
6. G. W. Moore, “Attractors and arithmetic,” [hep-th/9807056](#).
7. B. Pioline, “Lectures on on black holes, topological strings and quantum attractors,” *Class. Quant. Grav.* **23** (2006) S981, [hep-th/0607227](#).
8. P. Kraus, “Lectures on black holes and the AdS(3)/CFT(2) correspondence,” [hep-th/0609074](#).
9. B. de Wit, P. G. Lauwers, and A. Van Proeyen, “LAGRANGIANS OF N=2 SUPERGRAVITY - MATTER SYSTEMS,” *Nucl. Phys.* **B255** (1985) 569.
10. A. Strominger, “SPECIAL GEOMETRY,” *Commun. Math. Phys.* **133** (1990) 163–180.
11. T. Mohaupt, “New developments in special geometry,” [hep-th/0602171](#).
12. J. Bagger and E. Witten, “MATTER COUPLINGS IN N=2 SUPERGRAVITY,” *Nucl. Phys.* **B222** (1983) 1.
13. T. Mohaupt, “Black hole entropy, special geometry and strings,” *Fortsch. Phys.* **49** (2001) 3–161, [hep-th/0007195](#).
14. A. Ceresole, R. D’Auria, and S. Ferrara, “The Symplectic Structure of N=2 Supergravity and its Central Extension,” *Nucl. Phys. Proc. Suppl.* **46** (1996) 67–74, [hep-th/9509160](#).
15. T. Mohaupt, “Black holes in supergravity and string theory,” *Class. Quant. Grav.* **17** (2000) 3429–3482, [hep-th/0004098](#).
16. A. W. Peet, “TASI lectures on black holes in string theory,” [hep-th/0008241](#).
17. G. W. Gibbons and C. M. Hull, “A Bogomolny bound for general relativity and solitons in N=2 supergravity,” *Phys. Lett.* **B109** (1982) 190.
18. G. W. Gibbons and P. K. Townsend, “Vacuum interpolation in supergravity via super p-branes,” *Phys. Rev. Lett.* **71** (1993) 3754–3757, [hep-th/9307049](#).
19. S. Ferrara, R. Kallosh, and A. Strominger, “N=2 extremal black holes,” *Phys. Rev.* **D52** (1995) 5412–5416, [hep-th/9508072](#).
20. A. Strominger, “Macroscopic Entropy of N = 2 Extremal Black Holes,” *Phys. Lett.* **B383** (1996) 39–43, [hep-th/9602111](#).
21. S. Ferrara and R. Kallosh, “Universality of Supersymmetric Attractors,” *Phys. Rev.* **D54** (1996) 1525–1534, [hep-th/9603090](#).
22. F. Denef and G. W. Moore, “Split States, Entropy Enigmas, Holes and Halos,” [hep-th/0702146](#).
23. A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” *JHEP* **09** (2005) 038, [hep-th/0506177](#).
24. D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, “Rotating attractors,” *JHEP* **10** (2006) 058, [hep-th/0606244](#).
25. A. Dabholkar, A. Sen, and S. P. Trivedi, “Black hole microstates and attractor without supersymmetry,” *JHEP* **01** (2007) 096, [hep-th/0611143](#).
26. R. Kallosh, “New attractors,” *JHEP* **12** (2005) 022, [hep-th/0510024](#).
27. R. Kallosh, N. Sivanandam, and M. Soroush, “The non-BPS black hole attractor equation,” *JHEP* **03** (2006) 060, [hep-th/0602005](#).
28. R. Kallosh, N. Sivanandam, and M. Soroush, “Exact attractive non-BPS STU black holes,” *Phys. Rev.* **D74** (2006) 065008, [hep-th/0606263](#).
29. B. Sahoo and A. Sen, “Higher derivative corrections to non-supersymmetric extremal black holes in N = 2 supergravity,” *JHEP* **09** (2006) 029, [hep-th/0603149](#).
30. M. Shmakova, “Calabi-Yau black holes,” *Phys. Rev.* **D56** (1997) 540–544, [hep-th/9612076](#).

31. K. Behrndt, D. Lust, and W. A. Sabra, “Stationary solutions of  $N = 2$  supergravity,” *Nucl. Phys.* **B510** (1998) 264–288, [hep-th/9705169](#).
32. W. A. Sabra, “Black holes in  $N = 2$  supergravity theories and harmonic functions,” *Nucl. Phys.* **B510** (1998) 247–263, [hep-th/9704147](#).
33. W. A. Sabra, “General static  $N = 2$  black holes,” *Mod. Phys. Lett.* **A12** (1997) 2585–2590, [hep-th/9703101](#).
34. S. Ferrara, G. W. Gibbons, and R. Kallosh, “Black holes and critical points in moduli space,” *Nucl. Phys.* **B500** (1997) 75–93, [hep-th/9702103](#).
35. F. Denef, “Supergravity flows and D-brane stability,” *JHEP* **08** (2000) 050, [hep-th/0005049](#).
36. F. Denef, B. R. Greene, and M. Raugas, “Split attractor flows and the spectrum of BPS D-branes on the quintic,” *JHEP* **05** (2001) 012, [hep-th/0101135](#).
37. D. Gaiotto, A. Strominger, and X. Yin, “5D black rings and 4D black holes,” *JHEP* **02** (2006) 023, [hep-th/0504126](#).
38. I. Bena, P. Kraus, and N. P. Warner, “Black rings in Taub-NUT,” *Phys. Rev.* **D72** (2005) 084019, [hep-th/0504142](#).
39. R. Dijkgraaf, R. Gopakumar, H. Ooguri, and C. Vafa, “Baby universes in string theory,” *Phys. Rev.* **D73** (2006) 066002, [hep-th/0504221](#).
40. L. Alvarez-Gaume and S. F. Hassan, “Introduction to S-duality in  $N = 2$  supersymmetric gauge theories: A pedagogical review of the work of Seiberg and Witten,” *Fortsch. Phys.* **45** (1997) 159–236, [hep-th/9701069](#).
41. R. M. Wald, “Black hole entropy in the Noether charge,” *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038](#).
42. T. Jacobson, G. Kang, and R. C. Myers, “On black hole entropy,” *Phys. Rev.* **D49** (1994) 6587–6598, [gr-qc/9312023](#).
43. K. Behrndt *et al.*, “Classical and quantum  $N = 2$  supersymmetric black holes,” *Nucl. Phys.* **B488** (1997) 236–260, [hep-th/9610105](#).
44. G. Lopes Cardoso, B. de Wit, and T. Mohaupt, “Corrections to macroscopic supersymmetric black-hole entropy,” *Phys. Lett.* **B451** (1999) 309–316, [hep-th/9812082](#).
45. G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, “Stationary BPS solutions in  $N = 2$  supergravity with  $R^{*2}$  interactions,” *JHEP* **12** (2000) 019, [hep-th/0009234](#).
46. G. Lopes Cardoso, B. de Wit, and T. Mohaupt, “Deviations from the area law for supersymmetric black holes,” *Fortsch. Phys.* **48** (2000) 49–64, [hep-th/9904005](#).
47. G. Lopes Cardoso, B. de Wit, and T. Mohaupt, “Area law corrections from state counting and supergravity,” *Class. Quant. Grav.* **17** (2000) 1007–1015, [hep-th/9910179](#).
48. J. M. Maldacena, A. Strominger, and E. Witten, “Black hole entropy in M-theory,” *JHEP* **12** (1997) 002, [hep-th/9711053](#).
49. M. Guica and A. Strominger, “Wrapped  $M2/M5$  duality,” [hep-th/0701011](#).
50. E. Witten, “Constraints on Supersymmetry Breaking,” *Nucl. Phys.* **B202** (1982) 253.
51. J. M. Maldacena, G. W. Moore, and A. Strominger, “Counting BPS black holes in toroidal type II string theory,” [hep-th/9903163](#).
52. D. Gaiotto, A. Strominger, and X. Yin, “The  $M5$ -brane elliptic genus: Modularity and BPS states,” [hep-th/0607010](#).
53. R. Minasian, G. W. Moore, and D. Tsimpis, “Calabi-Yau black holes and  $(0,4)$  sigma models,” *Commun. Math. Phys.* **209** (2000) 325–352, [hep-th/9904217](#).
54. E. Witten, “Mirror manifolds and topological field theory,” [hep-th/9112056](#).
55. M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun. Math. Phys.* **165** (1994) 311–428, [hep-th/9309140](#).
56. A. Neitzke and C. Vafa, “Topological strings and their physical applications,” [hep-th/0410178](#).

57. M. Vonk, “A mini-course on topological strings,” [hep-th/0504147](#).
58. M. Marino, “Les Houches lectures on matrix models and topological strings,” [hep-th/0410165](#).
59. M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Holomorphic anomalies in topological field theories,” *Nucl. Phys.* **B405** (1993) 279–304, [hep-th/9302103](#).
60. I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, “Topological amplitudes in string theory,” *Nucl. Phys.* **B413** (1994) 162–184, [hep-th/9307158](#).
61. G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, “Black hole partition functions and duality,” *JHEP* **03** (2006) 074, [hep-th/0601108](#).
62. C. Beasley *et al.*, “Why  $Z(\text{BH}) = -Z(\text{top})^{*2}$ ,” [hep-th/0608021](#).
63. J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. Verlinde, “A farey tail for attractor black holes,” *JHEP* **11** (2006) 024, [hep-th/0608059](#).
64. P. Kraus and F. Larsen, “Partition functions and elliptic genera from supergravity,” *JHEP* **01** (2007) 002, [hep-th/0607138](#).
65. D. Gaiotto, A. Strominger, and X. Yin, “From  $\text{AdS}(3)/\text{CFT}(2)$  to black holes / topological strings,” [hep-th/0602046](#).
66. M. Banados, C. Teitelboim, and J. Zanelli, “Black hole entropy and the dimensional continuation of the Gauss-Bonnet theorem,” *Phys. Rev. Lett.* **72** (1994) 957–960, [gr-qc/9309026](#).
67. J. Callan, Curtis G. and F. Wilczek, “On geometric entropy,” *Phys. Lett.* **B333** (1994) 55–61, [hep-th/9401072](#).
68. C. Beasley and E. Witten, “New instanton effects in string theory,” *JHEP* **02** (2006) 060, [hep-th/0512039](#).
69. A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029](#).
70. M. Aganagic, H. Ooguri, N. Saulina, and C. Vafa, “Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings,” *Nucl. Phys.* **B715** (2005) 304–348, [hep-th/0411280](#).
71. M. Aganagic, H. Ooguri, and T. Okuda, “Quantum entanglement of baby universes,” [hep-th/0612067](#).
72. A. Dabholkar, “Exact counting of black hole microstates,” *Phys. Rev. Lett.* **94** (2005) 241301, [hep-th/0409148](#).
73. A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, “Precision counting of small black holes,” *JHEP* **10** (2005) 096, [hep-th/0507014](#).
74. E. P. Verlinde, “Attractors and the holomorphic anomaly,” [hep-th/0412139](#).
75. D. Shih and X. Yin, “Exact black hole degeneracies and the topological string,” *JHEP* **04** (2006) 034, [hep-th/0508174](#).
76. D. Jafferis and J. Marsano, “A DK phase transition in q-deformed Yang-Mills on  $S^{*2}$  and topological strings,” [hep-th/0509004](#).